# 17.3 Mass and Energy

**SECTION OUTCOMES**

- Apply quantitatively the laws of conservation of mass and energy, using Einstein's mass-energy equivalence.
- Conduct thought experiments as a way of developing an abstract understanding of the physical world as it relates to mass increase when an object approaches the speed of light.

**KEY TERMS**

- rest mass
- relativistic mass
- total energy
- rest energy

In the last section, you read a discussion based on mathematical equations that explained why no object with mass can travel at or above the speed of light. The discussion probably left you wondering why. If, for example, a spacecraft is travelling at 0.999 $c$, what would prevent it from burning more fuel, exerting more reaction force, and increasing its speed up to $c$?

The fact that no amount of extra force will provide that last change in velocity is explained when you discover that the mass of the spacecraft is also increasing. Einstein showed that, just as time dilates and length contracts when an object approaches the speed of light, its mass increases according to the equation $m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$. In the equation, $m$ is the mass as measured by an observer who sees the object moving with speed $v$, and $m_o$ is the mass as measured by an observer at rest relative to the object. The mass, $m$, is sometimes called the **relativistic mass** and $m_o$ is known as the **rest mass**.

As the speed of the object increases, the value of the denominator $\sqrt{1 - \frac{v^2}{c^2}}$ decreases. As $v$ approaches $c$, the denominator approaches zero and the mass increases enormously. If $v$ approaches $c$, the mass would approach $\frac{m_o}{0}$, or infinite. The speed of an object, measured from any inertial frame of reference, therefore must be less than the speed of light through space.

### RELATIVISTIC MASS

Relativistic mass is the product of rest mass and gamma.

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

| Quantity | Symbol | SI unit |
|---|---|---|
| relativistic mass | $m$ | kg (kilograms) |
| rest mass | $m_o$ | kg (kilograms) |
| speed of the mass relative to observer | $v$ | $\frac{\text{m}}{\text{s}}$ (metres per second) |
| speed of light | $c$ | $\frac{\text{m}}{\text{s}}$ (metres per second) |

**Unit Analysis**

$$\text{kilograms} = \frac{\text{kilograms}}{\sqrt{1 - \frac{\left(\frac{\text{metres}}{\text{second}}\right)^2}{\left(\frac{\text{metres}}{\text{seconds}}\right)^2}}} = \text{kilograms} \qquad \text{kg} = \frac{\text{kg}}{\sqrt{1 - \frac{\left(\frac{\text{m}}{\text{s}}\right)^2}{\left(\frac{\text{m}}{\text{s}}\right)^2}}} = \text{kg}$$

## MODEL PROBLEM

### Relativistic Masses

**An electron has a rest mass of $9.11 \times 10^{-31}$ kg. In a detector, it behaves as if it has a mass of $12.55 \times 10^{-31}$ kg. How fast is that electron moving relative to the detector?**

> **PROBLEM TIP**
>
> In questions involving masses, the masses form a ratio. It does not matter, therefore, what the actual mass units are, as long as they are the same for both the rest mass ($m_o$) and the moving (relativistic) mass ($m$).

### Frame the Problem

- The *mass* of the object appears to be *much greater* to an *observer* in a frame of reference that is *moving at relativistic speeds* than it does to an observer in the *frame of reference of the object*.
- The amount of the *increase in mass* is determined by the ratio of the *object's speed* and the *speed of light*.

### Identify the Goal

Determine the speed, $v$, of the electron relative to the detector

### Variables and Constants

| Known | Implied | Unknown |
|---|---|---|
| $m_o = 9.11 \times 10^{-31}$ kg | $c = 3.00 \times 10^8 \frac{\text{m}}{\text{s}}$ | $v$ |
| $m = 12.55 \times 10^{-31}$ kg | | |

### Strategy

Use the equation that relates the relativistic mass, rest mass, and speed.

Solve the equation for speed.

### Calculations

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{m_o}{m}$$

$$1 - \frac{v^2}{c^2} = \left(\frac{m_o}{m}\right)^2$$

$$\frac{v^2}{c^2} = 1 - \left(\frac{m_o}{m}\right)^2$$

$$v^2 = c^2\left(1 - \left(\frac{m_o}{m}\right)^2\right)$$

$$v = c\sqrt{1 - \left(\frac{m_o}{m}\right)^2}$$

Substitute numerical values and solve.

$$v = \left(3.00 \times 10^8 \frac{\text{m}}{\text{s}}\right)\sqrt{1 - \left(\frac{9.11 \times 10^{-31}\ \text{kg}}{12.55 \times 10^{-31}\ \text{kg}}\right)^2}$$

$$v = \left(3.00 \times 10^8 \frac{\text{m}}{\text{s}}\right)\sqrt{1 - 0.52692}$$

$$v = \pm\left(3.00 \times 10^8 \frac{\text{m}}{\text{s}}\right)(0.68780)$$

$$v = \pm 2.0634 \times 10^8 \frac{\text{m}}{\text{s}}$$

$$v \cong \pm 2.06 \times 10^8 \frac{\text{m}}{\text{s}}$$

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Since the problem is asking only for relative speed, the negative root has no meaning. Choose the positive value.

The speed of the electron relative to the detector is $2.06 \times 10^8\ \frac{\text{m}}{\text{s}}$ or $0.688c$.

### Validate

Since there is an appreciable mass increase, the object must be moving at a relativistic speed.

## PRACTICE PROBLEMS

**7.** A speck of dust in space has a rest mass of 463 $\mu$g. If it is approaching Earth with a relative speed of $0.100c$, what will be its mass as measured in the Earth frame of reference? Remember, in questions involving masses, the masses form a ratio, so it does not matter what the actual mass units are, as long as they are the same for both the rest mass and the moving, or relativistic, mass.

**8.** A neutron is measured to have a mass of $1.71 \times 10^{-27}$ kg when travelling at $6.00 \times 10^7$ m/s. Determine its rest mass.

**9.** How fast should a particle be travelling relative to an experimenter in order to have a measured mass that is 20.00 times its rest mass?

## Where Is the Energy?

At the start of this section, you examined the relativistic effects that occur when a spacecraft is approaching the speed of light. The conclusion was that its increasing mass must prevent it from accelerating up to the speed of light. However, while the thrusters on the spacecraft are firing, force is being exerted over a displacement, indicating that work was being done on the spacecraft. You know that, at non-relativistic speeds, the work would increase the spacecraft's kinetic energy. At relativistic speeds, however, the speed and thus the kinetic energy increase can only be very small. What, then, is happening to the energy that the work is transferring to the spacecraft?

Einstein deduced that the increased mass represented the increased energy. He expressed it in the formula $E_k = mc^2 - m_o c^2$ or $E_k = (\Delta m)c^2$. As before, $m$ is the mass of the particle travelling at speed $v$, and $m_o$ is its rest mass. The expression $mc^2$ is known as the **total energy** of the particle, while $m_o c^2$ is the **rest energy** of the particle. Rearranging the previous equation leads to $mc^2 = m_o c^2 + E_k$. The total energy of the particle equals the rest energy of the particle plus its kinetic energy.

## TOTAL ENERGY

The total energy (relativistic mass times the square of the speed of light) of an object is the sum of the rest energy (rest mass times the square of the speed of light) and its kinetic energy.

$$mc^2 = m_o c^2 + E_k$$

| Quantity | Symbol | SI unit |
|---|---|---|
| relativistic mass | $m$ | kg (kilograms) |
| rest mass | $m_o$ | kg (kilograms) |
| speed of light | $c$ | $\frac{m}{s}$ (metres per second) |
| kinetic energy | $E_k$ | J (joules) |

**Unit Analysis**

$$\text{kilogram}\left(\frac{\text{metres}}{\text{second}}\right)^2 = \text{kilogram}\left(\frac{\text{metres}}{\text{second}}\right)^2 + \text{joule} = \text{joule}$$

$$\text{kg}\left(\frac{\text{m}}{\text{s}}\right)^2 = \text{kg}\left(\frac{\text{m}}{\text{s}}\right)^2 + \text{J} = \text{J}$$

No wonder physicists had difficulty accepting Einstein's theory! In these equations, he is saying that mass and energy are basically the same thing and that the conversion factor relating them is $c^2$, the square of the speed of light. At the time that Einstein published his work, such changes in mass could not be measured. Eventually, with the advent of high-energy physics, these measurements have become possible.

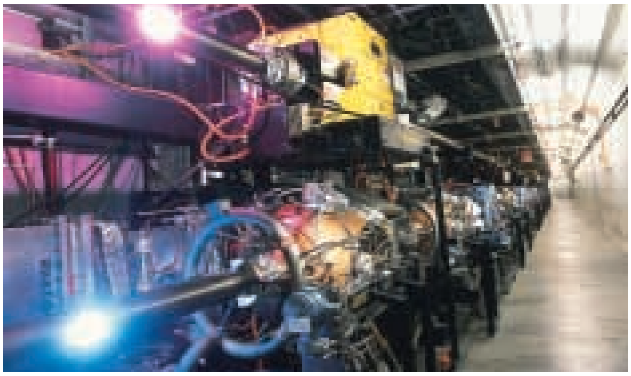

**Figure 17.13** In particle accelerators, such as this one at the Stanford Linear Accelerator Center in California, particles are accelerated to speeds very close to the speed of light. Their masses are measured and are found to agree with Einstein's prediction.

## MODEL PROBLEMS

### Kinetic Energy in a Rocket and in a Test Tube

**1. A rocket car with a mass of $2.00 \times 10^3$ kg is accelerated to $1.00 \times 10^8$ m/s. Calculate its kinetic energy**

**(a) using the classical or general equation for kinetic energy**

**(b) using the relativistic equation for kinetic energy**

### Frame the Problem

- The *classical* equation for *kinetic energy* is directly related to the object's *mass* and the *square of its velocity*.
- The *relativistic equation* for *kinetic energy* takes into account the concept that the *object's mass changes* with its *velocity* and accounts for this *relativistic mass*.

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## Identify the Goal

The kinetic energy, $E_k$, of the rocket car, using the classical expression for kinetic energy and then the relativistic expression for kinetic energy

## Variables and Constants

| Known | Implied | Unknown |
| --- | --- | --- |
| $m_o = 2.00 \times 10^3$ kg | $c = 3.00 \times 10^8 \frac{m}{s}$ | $m$ |
| $v = 1.00 \times 10^8 \frac{m}{s}$ | | $E_k$ |

## Strategy

Select the classical equation for kinetic energy. Substitute into the equation. Solve.

## Calculations

**(a)** $E_k = \frac{1}{2}mv^2$

$$E_k = \frac{1}{2}(2.00 \times 10^3 \text{ kg})\left(1.00 \times 10^8 \frac{\text{m}}{\text{s}}\right)^2$$

$$E_k = 1.00 \times 10^{19} \text{ J}$$

Calculate gamma.

**(b)** $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$\gamma = \frac{1}{\sqrt{1 - \frac{(1.00 \times 10^8 \frac{\text{m}}{\text{s}})^2}{(3.00 \times 10^8 \frac{\text{m}}{\text{s}})^2}}}$$

$$\gamma = 1.061$$

Select the relativistic equation for $E_k$. Rearrange in terms of gamma.

$$E_k = mc^2 - m_oc^2$$

$$E_k = c^2\left(m_o\gamma - m_o\right)$$

$$E_k = m_oc^2\left(\gamma - 1\right)$$

Substitute into the equation.

$$E_k = \left(2.00 \times 10^3 \text{ kg}\right)\left(3.00 \times 10^8 \frac{\text{m}}{\text{s}}\right)^2\left(1.061 - 1\right)$$

Solve the equation.

$$E_k = 1.09 \times 10^{19} \text{ J}$$

**(a)** The classical expression for kinetic energy yields $1.00 \times 10^{19}$ J.

**(b)** The relativistic expression for kinetic energy yields $1.09 \times 10^{19}$ J.

## Validate

Since gamma is close to one, a speed of $1.00 \times 10^8$ m/s does not provide a high degree of relativistic difference, so the two kinetic energies should not be too far apart.

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**2. A certain chemical reaction requires 13.8 J of thermal energy.**

**(a) What mass gain does this represent?**

**(b) Why would the chemist still believe in the law of conservation of mass?**

## Frame the Problem

- *Thermal energy* is the *kinetic energy* of *molecules*.

- Einstein's equation for *relativistic kinetic energy*, which represents the difference between the *total energy* and the *rest energy*, applies to the motion of molecules as well as to rockets.
- If *thermal energy* seems to *disappear* during a chemical reaction, it must have been *converted into mass*.

### Identify the Goal

The mass gain, $\Delta m$, during an absorption of 13.8 J of energy

### Variables and Constants

| Known | Implied | Unknown |
|---|---|---|
| $E_k = 13.8$ J | $c = 3.00 \times 10^8 \frac{m}{s}$ | $\Delta m$ |

### Strategy

Select the equation linking kinetic energy and mass.

$$E_k = \Delta mc^2$$

Rearrange to give the mass change.

$$\Delta m = \frac{E_k}{c^2}$$

Solve.

$$\Delta m = \frac{13.8 \text{ J}}{\left(3.00 \times 10^8 \frac{m}{s}\right)^2}$$

$$\Delta m = 1.533 \times 10^{-16} \text{ kg}$$

$$\Delta m \cong 1.53 \times 10^{-16} \text{ kg}$$

**(a)** The gain in mass is $1.53 \times 10^{-16}$ kg.

**(b)** This mass change is too small for a chemist to measure with a balance, so the total mass of the products would appear to be the same as the total mass of the reactants.

### Validate

The mass change at non-relativistic speeds should be extremely small.
**Note:** The source of the energy that is released during a chemical reaction is a loss of mass.

## PRACTICE PROBLEMS

**10.** A physicist measures the mass of a speeding proton as being $2.20 \times 10^{-27}$ kg. If its rest mass is $1.68 \times 10^{-27}$ kg, how much kinetic energy does the proton possess?

**11.** A neutron has a rest mass of $1.68 \times 10^{-27}$ kg. How much kinetic energy would it possess if it was travelling at $0.800c$?

**12.** How fast must a neutron be travelling relative to a detector in order to have a measured kinetic energy that is equal to its rest energy? Express your answer to two significant digits.

**13.** How much energy would be required to produce a kaon particle ($\kappa$) at rest with a rest mass of $8.79 \times 10^{-28}$ kg?

**14.** If an electron and a positron (antielectron), each with a rest mass of $9.11 \times 10^{-31}$ kg, met and annihilated each other, how much radiant energy would be produced? (In such a reaction involving matter and antimatter, the mass is completely converted into energy in the form of gamma rays.) Assume that the particles were barely moving before the reaction.

**15.** If the mass loss during a nuclear reaction is 14 $\mu$g, how much energy is released?

**16.** The Sun radiates away energy at the rate of $3.9 \times 10^{26}$ W. At what rate is it losing mass due to this radiation?

## Relativistic and Classical Kinetic Energy

It might seem odd that there are two apparently different equations for kinetic energy.

- At relativistic speeds, $E_k = mc^2 - m_o c^2$.
- At low (classical) speeds, $E_k = \frac{1}{2}mv^2$.

These equations are not as different as they appear, however. The first equation expands as follows.

$$E_k = mc^2 - m_o c^2$$
$$E_k = c^2(m - m_o)$$
$$E_k = c^2\left(\frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} - m_o\right)$$
$$E_k = m_o c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right]$$

This expression can be simplified by using an advanced mathematical approximation that reduces to the following when $v << c$.

$$E_k = c^2 m_o\left[1 - (-\tfrac{1}{2})\frac{v^2}{c^2} - 1\right]$$
$$E_k = \frac{1}{2}m_o v^2$$

The relativistic expression for kinetic energy therefore becomes the classical expression for kinetic energy at normal speeds.

You have now examined the basics of the special theory of relativity and have seen how the measurement of time, length, and mass depends on the inertial frame of reference of the observer. You have also seen that mass and energy are equivalent, that matter could be considered as a condensed form of energy. What happens, though, when the frame of reference is not inertial? Such considerations are the subject of Einstein's general theory of relativity, which deals with gravitation and curved space — concepts that are beyond the scope of this course.

## 17.3 Section Review

1. K/U What do the terms "total energy" and "rest energy" mean?
2. K/U What term represents the lowest possible mass for an object?
3. C Using the equations involved in relativity, give two reasons why the speed of light is an unattainable speed for any material object.
4. C Imagine that the speed of light was about 400 m/s. Describe three effects that would be seen in everyday life due to relativistic effects.
5. I How might an experimenter demonstrate that high-speed (relativistic) particles have greater mass than when they are travelling at a slower speed? Assume that the experimenter has some way of measuring the speeds of these particles.
6. MC **(a)** What must be true about the masses of the reactants and products for a combustion reaction? Why?
   **(b)** Why would a chemist never notice the effect in part (a)?