13.2 Motion Under a Linear Restoring Force

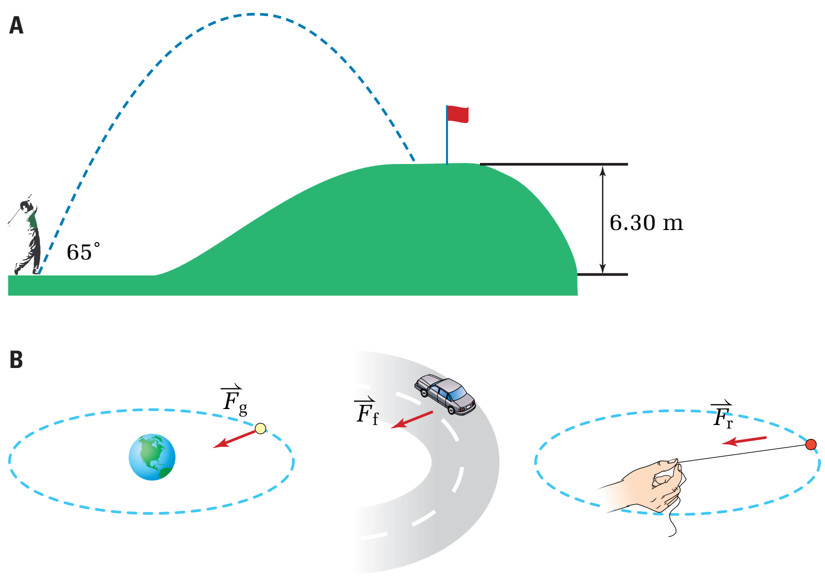


Figure 13.9 **(A)** Since the only force affecting the motion of the golf ball after it is hit is the constant force of gravity, the ball follows a parabolic trajectory. **(B)** Since the only force that is affecting the motion of the moon, the car, and the ball on a string is always perpendicular to their motion, they follow a circular path.

SECTION OUTCOMES

- Explain the relationship between displacement, velocity, time and acceleration for simple harmonic motion.
- Generate graphs for each variable in simple harmonic motion against time.

In Chapter 11, you learned that objects that are influenced only by a force that is constant and always in the same direction — such as the force of gravity near Earth's surface — will follow the path of a parabola. If the only force affecting an object's motion is always perpendicular to the direction of the motion, the object will follow a circular path. How can you describe, mathematically, the path of an object under a linear restoring force?

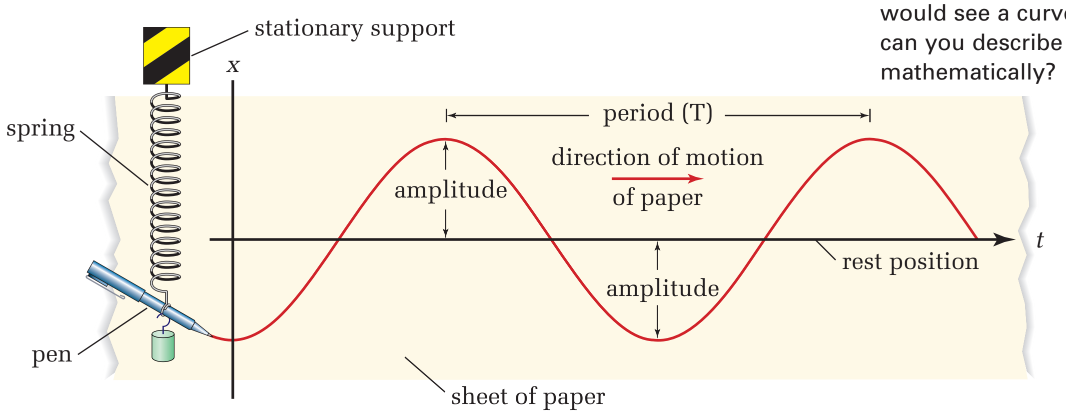


Figure 13.10 If you could attach a pen to a mass that is oscillating on the end of a spring and pull a piece of paper under the pen at a constant rate, you would see a curve like this. How can you describe this curve mathematically?

INVESTIGATION 13-C

Position, Velocity, and Acceleration and Simple Harmonic Motion

TARGET SKILLS
- Identifying variables
- Performing and recording
- Predicting

Before deriving equations that describe the properties of simple harmonic motion, having a sense of what to expect will make the equations seem more real. As you collect and analyze the motion data, try to develop an overview of the motion so that you will not get bogged down in the details.

Problem

What are the relationships among position, velocity, acceleration, and time in simple harmonic motion?

Equipment

- retort stand
- clamp
- spring
- masses
- motion sensor
- metre stick

Procedure

1. Before beginning to collect data, examine the shape of the curve in Figure 13.10. Make a rough sketch of one cycle of motion. From your analysis of the figure, predict the shape of the velocity versus time and acceleration versus time graphs.
2. Decide how best to set up the motion sensor to accurately detect the motion of the mass on the spring. When all members of your group have agreed on the set up, proceed to assemble it.
3. When you stretch the spring, be sure to avoid stretching it beyond its elastic limit. Test the system to find an amount of mass and an amplitude that will give easily measurable, smooth motion.
4. Collect data that covers at least one full cycle.
5. Either calculate by hand or with computer software, data that will allow you to plot three graphs, position versus time, velocity verses time, and acceleration verses time.
6. Plot, either by hand or by computer graphics, the graphs of position, velocity, and acceleration versus time.

Analyze and Conclude

1. How close did your predictions of the velocity and acceleration graphs come to the actual graph.
2. Select at least three different times along the graphs and describe the position, velocity, and acceleration of the mass at those times. Explain how the three curves are related to each other. For example, if the mass is approaching its equilibrium position, what is happening to the velocity and acceleration?
3. How do you think a spring with a larger force constant would affect the graphs?
4. How would a smaller mass change the graphs?
5. If possible, test your responses for the last two questions.

Position versus Time for Simple Harmonic Motion

The similarity of the motion of a mass on a spring to the motion of a mass moving on the edge of a disk that is rotating at a constant rate is an excellent tool to for developing an equation for a mass moving with simple harmonic motion. Study Figure 13.11 to see the relationship between points on the curve and the position of the mass on the rotating disk.

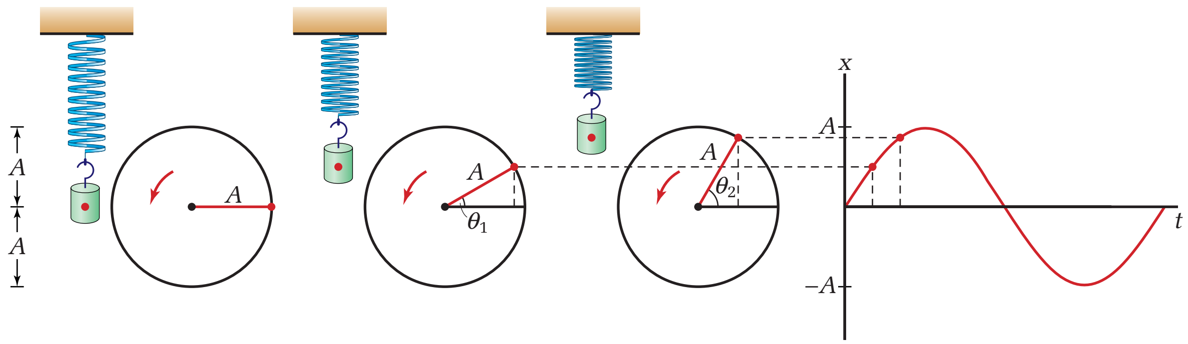


Figure 13.11 As the mass on the spring oscillates between a distance A above equilibrium to a distance A below equilibrium, the marker on the disk makes a complete circle.

To understand how the curve in figure 13.11 can be obtained, visualize the disk turning at a constant rate. The disk will turn through equal angles in equal time intervals so if you take a mental picture of the disk at equal time intervals, you will see where the location of the marker on the disk. The value of x on the graph is the same as the vertical distance of the marker on the disk from its centre line. The steps below will show you how to write a mathematical relationship for x.

- Write the definition of the sine of an angle.

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

- In the triangles formed on the disk by the radius A from the centre to the marker, the horizontal line through the centre, and the vertical line to the marker (which is equal to x), the "opposite side" is always the distance x, and the "hypotenuse" is always the radius, A.
  Therefore, you can write.

$$\sin\theta = \frac{x}{A}$$

- Solve for x.

$$x = A\sin\theta$$

- To write the angle θ in terms of time, recognize that the disk turns one full circle, or 360°, in one period. You can then write the rotation rate as shown.

$$\text{Rate of rotation} = \frac{360^\circ}{T}$$

- The angle θ at any time t is the rotation rate times the time.

$$\theta = \frac{360^\circ}{T}t$$

- Write the equation for x in terms of t by substituting the value in the last step for θ.

$$x = A\sin\frac{360^\circ}{T}t$$

- (Optional) For those of you who are familiar with radian measure for angles, you know that 360° is equal to 2π radian. You will often see the equation for x as shown.

$$x = A\sin\frac{2\pi}{T}t$$

- (Optional) Since the frequency, f, is the inverse of the period, T, another familiar form of the equation is shown.

$$x = A\sin 2\pi f\, t$$

Since the equation relating the position, x, and time, t, involves a sine function, it is called a sine wave. All objects in simple harmonic motion follow the path of a sine wave.

Velocity and Acceleration versus Time for Simple Harmonic Motion

If you can find an equation that relates the velocity of an object in simple harmonic motion to the position, you can use that equation to write the velocity as a function of time. You used such an equation when developing the equation for the period of oscillation of a mass and spring system. Starting with the expression for the total energy of an object in simple harmonic motion (on page 601) you derived the equation, $v^2 = \frac{k}{m}(A^2 - x^2)$. The following steps will show you how to use this equation to develop an expression for the velocity of a simple harmonic oscillator in terms of time.

- Substitute the equation for the position in terms of time into the equation for velocity and simplify.

$$v^2 = \frac{k}{m}(A^2 - x^2)$$

$$v^2 = \frac{k}{m}\left[A^2 - \left(A\sin\frac{360^\circ}{T}t\right)^2\right]$$

$$v^2 = \frac{k}{m}\left(A^2 - A^2\sin^2\frac{360^\circ}{T}t\right)$$

- Factor A^2 out of the brackets. $$v^2 = \frac{k}{m}A^2\left(1 - \sin^2 \frac{360^\circ}{T}t\right)$$
- If you have had a math course in which you learned trigonometric identities, you will know that $1 - \sin^2\theta = \cos^2\theta$ for any angle. A proof is also given in the Math Link. Replace the expression in brackets with the square of the cosine of the angle. Then take the square root of both sides of the equation to find the equation for the velocity of a harmonic oscillator. $$v^2 = \frac{k}{m}A^2 \cos^2 \frac{360^\circ}{T}t$$ $$v = A\sqrt{\frac{k}{m}} \cos \frac{360^\circ}{T}t$$

The velocity of a harmonic oscillator in terms of time follows a cosine wave as shown in Figure 13.12.

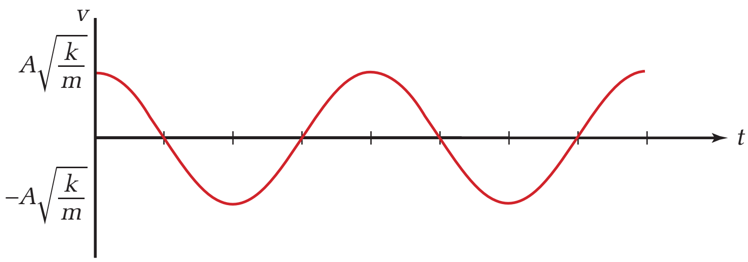
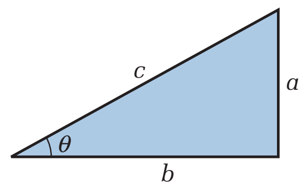


Figure 13.12 Notice that the cosine wave for velocity of a simple harmonic oscillator has a shape identical to the sine wave but it is translated horizontally by one fourth of the period.

If you can find an equation that relates the acceleration of a simple harmonic oscillator to its position, you can develop an equation that relates the acceleration to time using the same approach that you used for velocity. Since you know that acceleration is always related to force by Newton's second law, consider the equation for the restoring force of a simple harmonic oscillator. Use the equation to develop the relationship between acceleration and time for a simple harmonic oscillator.

- Write the equation for the linear restoring force. $$F = -kx$$
- If the linear restoring force is the only force affecting the motion of the simple harmonic oscillator, you can set the restoring force equal to the force in Newton's second law. Then solve for the acceleration, a. $$F = ma$$ $$ma = -kx$$ $$a = -\frac{k}{m}x$$

Math Link



The trigonometric identity, $\sin^2\theta + \cos^2\theta = 1$ or $\cos^2\theta = 1 - \sin^2\theta$ is directly related to the Pythagorean theorem as shown here. First, write the equation for the Pythagorean theorem for the triangle above.

$$a^2 + b^2 = c^2$$

Divide both sides of the equation by c^2 and simplify.

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

Write the expressions for $\sin\theta$ and $\cos\theta$ for the same triangle and substitute them in for the equivalent ratios in the equation above.

$$\sin\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c}$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

The result is true for any angle θ.

- Substitute the equation relating position and time for the simple harmonic oscillator into the equation above and you have the expression that describes the acceleration as a function of time.

$$a = -A\left(\frac{k}{m}\right)\sin\frac{360^\circ}{T}t$$

The acceleration of a simple harmonic oscillator follows the negative of a sine wave as shown in figure 13.13.

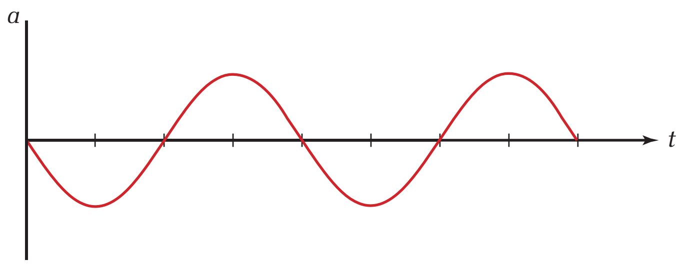


Figure 13.13 Once again, the negative of the sine wave, which describes the acceleration of a simple harmonic oscillator has a shape identical to the sine wave but it is inverted.

QUICK LAB

Graphing the Motion of a Simple Harmonic Oscillator

TARGET SKILLS
- **Identifying variables**
- **Analyzing and interpreting**

Graphing data makes curves more real than just looking at figures. In this activity, you will calculate and graph the motion of a mass oscillating on the end of a spring. Imagine that you have attached a 2.19 kg mass to the end of a spring with a force constant of 15.0 N/m. You stretch the spring a distance of 10 cm from its equilibrium position and it begins to oscillate. First, calculate the period with which it will oscillate. Then make a table with the headings, time, position, velocity, and acceleration. Make 24 rows and in the first column list 0.1 s, 0.2 s, 0.3 s up to the time of one period. Using the equations that you derived above, calculate the position, velocity, and acceleration for each time. (Hint: if you can find the symmetry of the shape of sine and cosine waves, you will only have to do calculations for one fourth of the data points.) On a large piece of graph paper, plot one period of position versus time. Just below the first graph, plot velocity versus time using the same scale for time. Finally, below the second graph, plot acceleration versus time on the same time scale. Now you can compare the position, velocity, and acceleration of the mass at any time.

Analyze and Conclude

1. Locate the times when the position is at the equilibrium position. From the graphs, describe the velocity and acceleration at these times.
2. Locate the times when the position is at its maximum. From the graphs, describe the velocity and acceleration at these times.

At this point, simple harmonic motion may still seem very theoretical and artificial to you. However, as you progress with your studies of physics, you will discover more and more cases for which simple harmonic motion is a good model. In some situations, electrons in atoms can be modelled by simple harmonic oscillators. On the other end of the scale, buildings and even bridges, as you saw in Chapter 8, can vibrate. Engineers must understand simple harmonic motion in order to prevent excessive oscillations. Vibrations are part of everyday life.

13.2 Section Review

1. K/U Explain how the rotating disk was used to develop an equation that relates the position of a mass on a spring to time.
2. K/U Describe, qualitatively, the relationships among the curves for position, velocity, and acceleration versus time.
3. MC Do research to find out how simple harmonic motion models some practical situation.
4. I Design and, if possible, carry out an experiment that would produce periodic motion that was not simple harmonic motion. Explain how you might achieve this.
5. Review your graphs of simple harmonic motion from the Quick Lab on the previous page. Are there cases where the velocity is zero but the acceleration is not zero? Explain why this can or cannot happen.
6. Are there cases where the acceleration is zero but the velocity is not zero? Explain why this can or cannot happen.
7. Examine the position versus time graph from your Quick Lab and, from your observations, roughly sketch a graph of elastic potential energy versus time.
8. Examine the velocity versus time graph and, from your observations, roughly sketch a graph of kinetic energy versus time.
9. Describe the relationship between the two energy versus time graphs.