13.1 Describing Periodic Motion

SECTION OUTCOMES

- Explain the relationship between potential energy and kinetic energies of a mass in simple harmonic motion.
- Solve problems relating to the period of simple harmonic motion for a mass on a spring and a pendulum.
- Compile and organize data using data tables and graphs to facilitate interpretation of the data.

KEY TERMS

- periodic motion
- simple harmonic motion

When you carried out Investigation 13-A, you stretched a spring to both measure its force constant and to turn it into a projectile. This activity encouraged you to recall the properties of elastic objects that you studied in Chapter 7 which focussed on the elastic potential energy stored in a stretched spring. In this chapter, you will direct your attention to the resulting motion that occurs when you release the mass and allow it to oscillate.

Defining Simple Harmonic Motion

Any motion that repeats itself precisely over equal periods of time is classed as periodic motion. If that **periodic motion** is generated by a linear restoring force, it is **simple harmonic motion**. All three of the graphs in Figure 13.1 represent periodic motion because the motion of an object along the *x* axis is repetitive but only graph C represents simple harmonic motion.

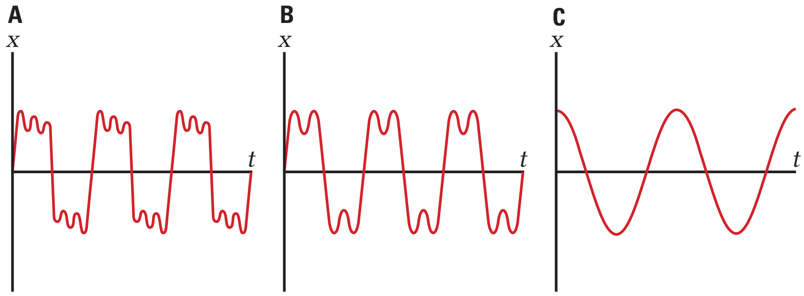

Figure 13.1 Periodic motion can appear very erratic (**A** and **B**) but simple harmonic motion (**C**) follows a smooth curve.

You probably recall from Chapter 6, that the mathematical statement of Hooke's law, $F = -kx$, is a linear equation because the force, F, is directly proportional to x, the amount of extension or compression of the spring. The negative sign indicates that the force always opposes the direction of the stretch, "restoring" the object to its rest or equilibrium position. Therefore, the force that a spring exerts on a mass causes the mass to oscillate with simple harmonic motion.

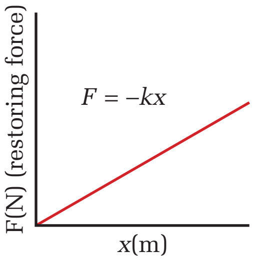

Figure 13.2 The curve of force versus amount of stretch or compression for a Hooke's law spring is linear. The slope of the line is the force constant.

The Period of a Mass on a Spring

Any characteristics that you can describe for the motion of a mass on a spring will provide a model for all types simple harmonic motion. Two of the most obvious and related quantities that describe any form of periodic motion are the period and the frequency. You learned in Chapter 8 that the period is the length of time required for an object to complete one full cycle of motion. The frequency is the inverse of the period and is the number of cycles that occur in a given time period. In SI units, the frequency is the number of cycles per second. You probably recall that the unit, "cycles per second," has been given the name hertz (Hz).

How can you predict the period or frequency of a mass oscillating on the end of a spring based on the properties of the system? Start with the relationships you already know that involve the mass, the force constant, and the amount of extension or compression of the spring? Figure 13.3 is similar to Figure 7.5 and will help you recall the energy relationships for the mass and spring system. You will use these relationships to develop an equation describing the period of a mass on a spring.

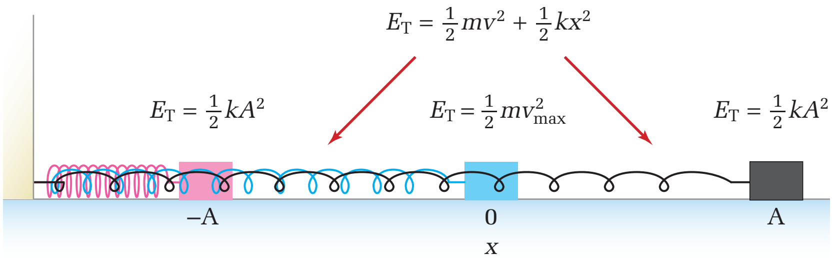

Figure 13.3 The mass, the force constant, and the amplitude of the oscillating spring determine the total energy of the spring.

At any position, x, the total energy of the mass and spring system is $E_T = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$. At either end of its oscillation, the mass stops and changes direction. Therefore, the velocity is zero so all of the energy is elastic potential energy, $E_T = \frac{1}{2}kA^2$. At equilibrium $(x = 0)$, the spring is not stretched so all of the energy is kinetic energy, $E_T = \frac{1}{2}mv^2_{max}$. In the following steps, you will use this information to derive an expression for the speed of the mass at any position of its motion. Later, you will see how this expression will lead you to an equation for the period of the motion of the mass.

PHYSICS FILE

When physicists develop the basic concepts behind a system or type of motion, they work with idealized conditions. One of these ideal conditions for a mass oscillating on the end of the spring is that the spring is "massless." In this case, massless means that the mass of the spring is so much smaller than the mass of the oscillating object that the mass of the spring has no affect on the motion.

- Because the two expressions on the right are both equal to the total energy of the spring and mass system, you can set them equal to each other.

$$E_T = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \text{ and } E_T = \frac{1}{2}kA^2$$
$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

- Multiply both sides of the equation by 2 and then solve for v^2.

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$
$$kA^2 = mv^2 + kx^2$$
$$mv^2 = kA^2 - kx^2$$
$$v^2 = \frac{kA^2 - kx^2}{m}$$

- Factor k out of the numerator and rewrite.

$$v^2 = \frac{k}{m}(A^2 - x^2)$$

- Move the A^2 outside of the brackets.

$$v^2 = \frac{k}{m}A^2\left(1 - \frac{x^2}{A^2}\right)$$

- To find a simpler expression for $\frac{k}{m}A^2$, set the two expressions shown here equal to each other and then solve for v^2_{max}.

$$E_T = \frac{1}{2}mv^2_{max} \text{ and } E_T = \frac{1}{2}kA^2$$
$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2_{max}$$
$$kA^2 = mv^2_{max}$$
$$v^2_{max} = \frac{k}{m}A^2$$

- Substitute v^2_{max} into the equation for v^2 because it is equal to $\frac{k}{m}A^2$.

$$v^2 = v^2_{max}\left(1 - \frac{x^2}{A^2}\right)$$

- Finally, solve for v by taking the square root of both sides of the equation.

$$v = v_{max}\sqrt{1 - \frac{x^2}{A^2}}$$

To use the last relationship in the development of an equation for the period of the motion of the mass on the spring, you will need to find another type of motion that has a velocity with an identical mathematical relationship. The motion that you need is the projection — or shadow — of a marker moving with uniform circular motion. This image can be obtained by placing a marker on the edge of a disk that is rotating at a constant speed as shown in Figure 13.4A.

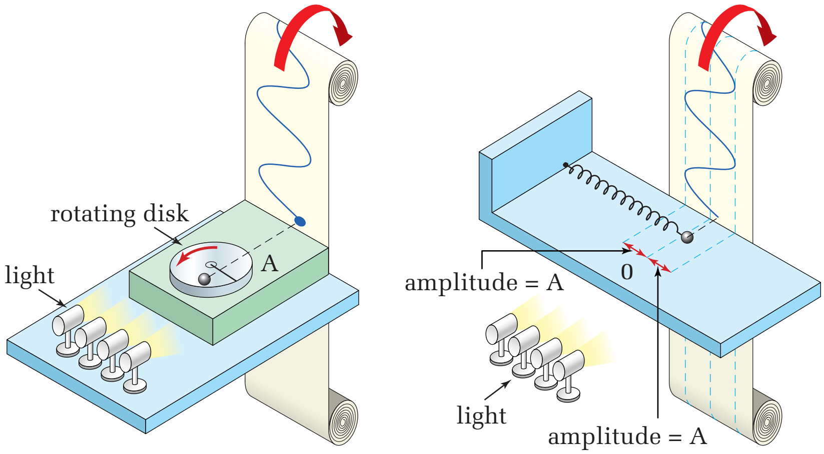

Figure 13.4 The shadows of (**A**) the marker on the edge of a rotating disk and of (**B**) a mass on the end of a spring are recorded on a tape that is moving at a constant speed.

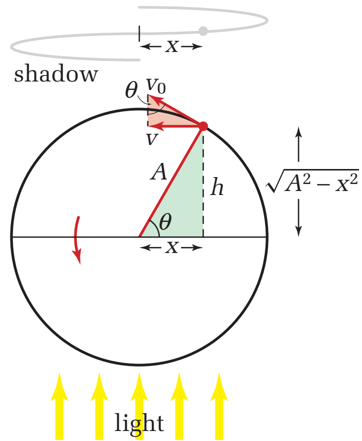

Figure 13.5 The disk of radius *A* is rotating counterclockwise at a constant speed.

The shadows of the marker on the rotating disk and mass on the oscillating spring appear to be identical but are they? To answer this question, study Figure 13.5 which is looking directly down from above the disk.

The radius of the rotating disk is A, making the amplitude of the motion of the shadow A. The following steps show you how to find an expression for side h of the green triangle.

- Apply the Pythagorean theorem. $$A^2 = x^2 + h^2$$
- Rearrange and solve for h. $$h^2 = A^2 - x^2$$ $$h = \sqrt{A^2 - x^2}$$

The vector, v_0, represents the tangential speed of the marker which is constant for uniform circular motion. The vector, v, is the speed of the shadow of the marker. Since vector v_0 is perpendicular to the radius of the circle, A, and vector v is perpendicular to side h of the triangle, the angles between these pairs of lines, labelled θ, must be the same making the triangles similar. Use these similar triangles to find an expression for the speed, v, of the shadow of the marker.

- The ratio of homologous sides of similar triangles are equal. $$\frac{v}{v_0} = \frac{\sqrt{A^2 - x^2}}{A}$$

  Rearrange. $$\frac{v}{v_0} = \sqrt{\frac{A^2 - x^2}{A^2}}$$ $$\frac{v}{v_0} = \sqrt{1 - \frac{x^2}{A^2}}$$
- Solve for v. $$v = v_0\sqrt{1 - \frac{x^2}{A^2}}$$

The expression for the speed of the shadow of the marker is mathematically identical to the expression for the speed of a mass on the end of an oscillating spring if v_0 is equal to v_{max}. This will be true if the amplitude of the mass on the spring is the same as the radius of the rotating disk. The similarity of these mathematical relationships demonstrates that the shadows of the marker on the rotating disk and the mass on the oscillating spring *are* identical. Any relationships derived for the shadow of the marker on the rotating disk are true for the mass on the spring. The following steps show you how to find the period of the shadow of the marker on the rotating disk.

- Write the general equation for speed or magnitude of velocity.

$$v = \frac{\Delta d}{\Delta t}$$

- Apply the equation for speed to the marker moving in a circle of radius A. The distance travelled in one period is the circumference of the circle.

$$\Delta d = 2\pi A \text{ and } \Delta t = T$$
$$v_0 = \frac{2\pi A}{T}$$

- Solve for the period, T.

$$T = \frac{2\pi A}{v_0}$$

- The v_0 in this equation is the same as v_{max} for the mass on the spring. Set v_0 equal to the expression for v_{max} in the derivation on page 601.

$$v_{max}^2 = \frac{k}{m}A^2$$
$$v_0^2 = \frac{k}{m}A^2$$
$$v_0 = A\sqrt{\frac{k}{m}}$$

- Substitute the value for v_0 into the equation for the period and simplify.

$$T = \frac{2\pi \cancel{A}}{\cancel{A}\sqrt{\frac{k}{m}}}$$
$$T = 2\pi\sqrt{\frac{m}{k}}$$

- You can use the equation for the period to derive an equation for the frequency of a mass on a spring.

$$f = \frac{1}{T}$$
$$f = \frac{1}{2\pi\sqrt{\frac{m}{k}}}$$
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

PERIOD OF MASS ON SPRING

The period of a mass that is oscillating on the end of a spring is two pi times the square root of the quotient of the mass and the force constant.

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Quantity	Symbol	SI unit
period	T	s (seconds)
mass	m	kg (kilograms)
force constant	k	$\frac{\text{N}}{\text{m}}$ (newtons per metre)

Unit Analysis

$$\text{seconds} = \sqrt{\frac{\text{kilograms}}{\frac{\text{newtons}}{\text{metre}}}}$$

$$\text{s} = \sqrt{\frac{\text{kg}}{\frac{\text{N}}{\text{m}}}} = \sqrt{\frac{\text{kg}\cdot\text{m}}{\text{N}}} = \sqrt{\frac{\cancel{\text{kg}\cdot\text{m}}}{\frac{\cancel{\text{kg}\cdot\text{m}}}{\text{s}^2}}} = \sqrt{\text{s}^2} = \text{s}$$

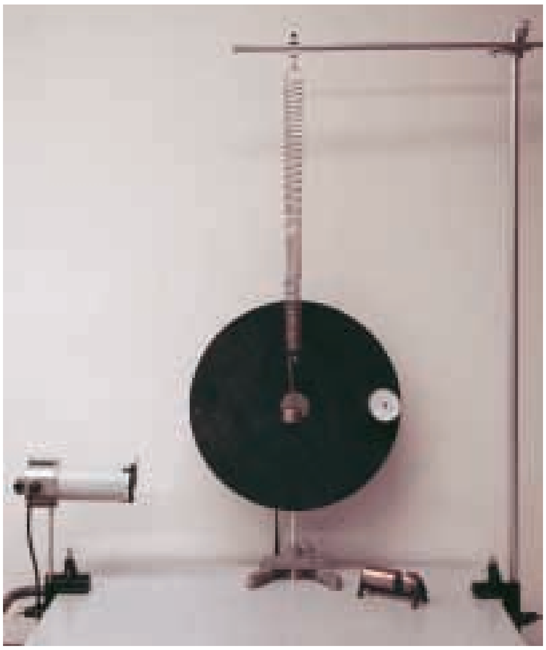

Figure 13.6 Your school might have an apparatus like this to demonstrate that the motion of the shadow of a mass on a spring is identical to the motion of a shadow of an object on the edge of a rotating disk. If not, your class might choose to make a project out of designing and building such an apparatus.

NASA PURSVE/UNM Physics And Astronomy Department Regener Hall Lecture Demonstratons

Knowing the relationship between the period of a mass and spring system in conjunction with your previous knowledge of the energy relationships of a mass and spring system allows you to solve a wider variety of problems.

Web Link

www.mcgrawhill.ca/links/atlphysics

Simulations of masses oscillating on springs, in which you can vary the mass and the force constant of the spring and observe the results, can be found on the Internet. To find these simulations, go to the Internet site above and follow the links.

MODEL PROBLEM

You stretch a spring a distance of 12.0 cm from its rest length and release it. A 125 g mass on the end of the spring completes exactly 20.0 cycles in 15.5 s. Find:

(a) the period

(b) the force constant of the spring

(c) the total energy of the system

(d) the maximum speed of the mass

(e) the speed of the mass when it is 10.0 cm from its equilibrium position

Frame the Problem

- When you stretch the spring, you give it *elastic potential energy* which is then the *total energy* of the system.
- When you release the mass, the elastic potential is transformed into *kinetic energy* and back to elastic potential energy, cyclically.
- At the moment that total energy is in the form of *kinetic energy*, the mass is at its *maximum speed*.
- The number of *cycles* completed in one *second* is the *period*.
- The *force constant* is related to the *period* and the *mass*.
- At any point in the motion of the mass, the *total energy* is equal to the *sum* of the *kinetic* and *potential energies*.
- Always convert all units to SI units before substituting values into equations.

Identify the Goal

(a) the period, T, of oscillation of the mass

(b) the force constant, k, of the spring

(c) the total energy, E_T, of the system

(d) the maximum speed, v_{max}, of the mass

(e) the speed, v, of the mass at $x = 10.0$ cm

Variables and Constants

Known	Unknown
$A = 12.0$ cm	T
$m = 125$ g	k
$N = 20.0$ (cycles)	E_T
$\Delta t = 15.5$ s	v_{max}
$x = 10.0$ cm	$v_{(at\ x\ =\ 10.0\ cm)}$

Strategy | Calculations

Find the period from the definition.

$$T = \frac{\Delta t}{N}$$
$$T = \frac{15.5\ \text{s}}{20.0}$$
$$T = 0.775\ \text{s}$$

(a) The period of the motion is 0.775 s.

Write the equation that relates the period to the mass and the force constant then square both sides of the equation.

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$T^2 = 4\pi^2\left(\frac{m}{k}\right)$$

Solve the equation for the force constant.

$$(\cancel{T^2})\left(\frac{k}{\cancel{T^2}}\right) = \left[4\pi^2\left(\frac{m}{\cancel{k}}\right)\right]\frac{\cancel{k}}{T^2}$$
$$k = 4\pi^2\left(\frac{m}{T^2}\right)$$

Substitute numerical values and solve.

$$k = 4\pi^2\left[\frac{0.125\ \text{kg}}{(15.5\ \text{s})^2}\right]$$
$$k = 0.02054\ \frac{\text{kg}}{\text{s}^2}$$

Prove that a kg/s^2 is equivalent to a N/m by multiplying the numerator and denominator of the units by *m*.

$$\frac{\text{kg}}{\text{s}^2}\cdot\frac{\text{m}}{\text{m}} = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}\cdot\frac{1}{\text{m}} = \frac{\text{N}}{\text{m}}$$
$$k \cong 0.0205\ \frac{\text{N}}{\text{m}}$$

(b) The force constant of the spring is 0.0205 N/m.

Before the mass is released, the total energy of the system is the elastic potential energy.

$$E_\text{T} = \frac{1}{2}kA^2$$
$$E_\text{T} = \frac{1}{2}\left(0.02054\ \frac{\text{N}}{\text{m}}\right)(0.120\ \text{m})^2$$
$$E_\text{T} = 1.479 \times 10^{-4}\ \text{J}$$
$$E_\text{T} \cong 1.48 \times 10^{-4}\ \text{J}$$

(c) The total energy of the system is 1.48×10^{-4} J.

Write the equation kinetic energy for total energy.

$$E_\text{T} = \frac{1}{2}mv_\text{max}^2$$

Solve for maximum speed.

$$v_\text{max}^2 = \frac{2E_\text{T}}{m}$$
$$v_\text{max} = \sqrt{\frac{2E_\text{T}}{m}}$$

Substitute numerical values and solve.

$$v_\text{max} = \sqrt{\frac{2(1.479 \times 10^{-4}\ \text{J})}{0.125\ \text{kg}}}$$
$$v_\text{max} = \sqrt{2.3663 \times 10^{-3}\ \frac{\frac{\cancel{\text{kg}}\cdot\text{m}}{\text{s}^2}}{\cancel{\text{kg}}}}$$
$$v_\text{max} = 4.8646 \times 10^{-2}\ \frac{\text{m}}{\text{s}}$$
$$v_\text{max} \cong 4.86 \times 10^{-2}\ \frac{\text{m}}{\text{s}}$$

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(d) The maximum speed of the mass is 4.86×10^{-2} m/s.

Write the equation for total energy at any point in the motion of the mass. Solve the equation for the speed.

$$E_T = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\frac{1}{2}mv^2 = E_T - \frac{1}{2}kx^2$$

$$v^2 = \frac{2\left(E_T - \frac{1}{2}kx^2\right)}{m}$$

$$v = \sqrt{\frac{2\left(E_T - \frac{1}{2}kx^2\right)}{m}}$$

Substitute numerical values and solve.

$$v = \sqrt{\frac{2\left(E_T - \frac{1}{2}kx^2\right)}{m}}$$

$$v = \sqrt{\frac{2\left(1.479 \times 10^{-4}\ \mathrm{J} - \frac{1}{2}\left(0.02054\ \frac{\mathrm{N}}{\mathrm{m}}\right)(0.100\ \mathrm{m})^2\right)}{0.125\ \mathrm{kg}}}$$

$$\left[\text{Note: } \sqrt{\frac{\mathrm{J}}{\mathrm{kg}}} = \sqrt{\frac{\frac{\cancel{\mathrm{kg}}\cdot\mathrm{m}^2}{\mathrm{s}^2}}{\cancel{\mathrm{kg}}}} = \sqrt{\frac{\mathrm{m}^2}{\mathrm{s}^2}} = \frac{\mathrm{m}}{\mathrm{s}}\right]$$

$$v = \sqrt{7.232 \times 10^{-4}\ \frac{\mathrm{J}}{\mathrm{kg}}}$$

$$v = 2.6892 \times 10^{-2}\ \frac{\mathrm{m}}{\mathrm{s}}$$

$$v \cong 0.0269\ \frac{\mathrm{m}}{\mathrm{s}}$$

(e) The speed of the mass when it was 10.0 cm from equilibrium was 0.0269 m/s.

Validate

All of the units cancelled properly to give the correct units for every answer. The force constant and the maximum speed are very small but there is a relatively small mass oscillating very slowly so you would expect that the restoring force and the energy would be small. The average speed of the mass was 48 cm (equilibrium to one maximum at 12 cm, to the other maximum at –12 cm and back to equilibrium) in 15.5 s. That is a speed of $\frac{0.48\ \mathrm{m}}{15.5\ \mathrm{s}}$ or about $0.031\ \frac{\mathrm{m}}{\mathrm{s}}$. You would expect the maximum speed to be somewhat larger which it is.

PRACTICE PROBLEMS

1. A 0.525 kg mass oscillates on the end of a spring having a force constant of 85.0 N/m. What is the period of the oscillations?

2. You tied a 275 g ball to the end of an elastic band and stretched the band and released it. You counted 15 complete oscillations in 12 s. If the elastic band obeys Hooke's law, what is the force constant of the elastic band?

3. A 0.250 kg mass is oscillating on the end of a spring having a force constant of 154 N/m on a frictionless surface. The amplitude of the oscillations is 0.34 m.

(a) What is the period of the motion?

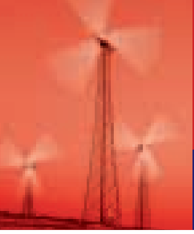
(b) What is the maximum speed of the mass?

(c) What is the speed of the mass when it is 0.16 m from its equilibrium position?

4. The maximum speed of a 1.45 kg block that is oscillating on the end of a spring is 0.84 m/s.

(a) If the amplitude of the oscillations is 12.0 cm, what is the force constant of the spring?

(b) What is the period of the motion?



INVESTIGATION 13-B

The Period of a Mass on a Spring

TARGET SKILLS
- Predicting
- Performing and recording
- Analyzing and interpreting

You just derived an equation for the period of a mass and spring system in which the mass was moving horizontally on a frictionless surface. Is the period of a mass hanging from a spring under the force of gravity the same as the calculated value of the period for the mass on a spring moving on a horizontal, frictionless surface?

Problem

Is the observed period of a mass oscillating on the end of a vertical spring the same as the period predicted by the equation, $T = 2\pi\sqrt{\frac{m}{k}}$.

Equipment

- retort stand
- spring
- hanger
- set of masses
- balance
- stopwatch

Procedure

1. Predict whether the period of a spring mass system that is vertical and affected by gravity will have the same period as the theoretical period developed for a horizontal spring and mass system.
2. Using the methods that you used in Investigation 13-A and Investigation 6-A, determine the force constant of the spring.
3. After you have completed measuring the force constant, hang a 100 g mass on the spring. If you are using a mass hanger, determine the exact total mass that the spring is supporting. Record the amount of mass.
4. Stretch the spring about 2 cm and release it. After one or two cycles, start the stopwatch and start counting cycles.
5. Determine the time for ten complete cycles. Divide the time by ten and record the period.
6. Stretch the spring about 3 cm and release it. (Note: Depending on the spring you are using, you might want to reduce the amplitude instead of increasing it in order to prevent the spring from being stretched beyond its elastic limit.) Repeat steps 3 and 4.
7. Add 50 g to the mass on the spring and record the total amount of mass on the spring.
8. Repeat steps 3, 4, and 5 for the larger mass.
9. Calculate the theoretical periods for the two different amounts of mass and the two different amplitudes from the masses and force constant.
10. Determine the percent deviation between your calculated and observed values for the period of the spring and mass systems.

Analyze and Conclude

1. Why is measuring the time for ten cycles then dividing by ten more accurate than measuring one cycle?
2. Compare the periods that were calculated using the different amplitudes. Was there a significant difference between the observed periods for one amount of mass and two different amplitudes? If so, give some possible explanations.
3. Were the percent deviations between the observed and calculated values for the period small enough to be explained by experimental or measurement error? What are some possible causes of experimental error?
4. From your results, state whether you believe that the period of mass and spring systems for hanging masses are the same as the theoretical periods. Explain your reasoning.
5. Was your original prediction correct? Explain your reasoning for your original prediction.

Period of a Simple Pendulum

Grandfather clocks, like the one in the photograph, are an attractive piece of history. Is the ornamental pendulum swinging simply for appearance? No, the pendulum is a precision scientific instrument. The carefully adjusted period of the pendulum determines the speed with which the hands on the clock turn and therefore make it possible for the clock to keep accurate time.

History Link

Galileo was the first person to have the idea of using a pendulum to control the timing of clocks. At the age of 77 and totally blind, Galileo described his idea to a pupil, Viviani and to his son, Vincenzio. His son drew diagrams of the proposed clock but neither Galileo nor Vincenzio lived long enough to test the design. Soon thereafter, Dutch scientist, Christiaan Huygens and British scientist, Robert Hooke, improved on the design and pendulum clocks became a reality. For about 300 years, pendulum clocks were the most dependable, accurate clocks available.

Figure 13.7 The position of the ornamental pendulum bob can be precisely adjusted to create an exact period so the clock will tell perfect time. The gravitational potential energy of the slowly dropping weights provides energy for the pendulum to continue to swing in spite of loss of energy due to friction.

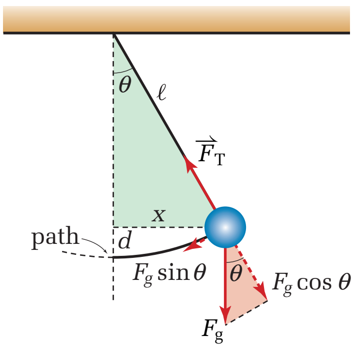


Figure 13.8 When the pendulum is drawn back so that the string makes an angle θ with the vertical equilibrium position, the gravitational force vector, F_g makes an angle θ with the line along the pendulum string. The angles must be the same because they are angles made when one straight line intersects with two vertical, therefore parallel, lines.

In this section, you will develop an expression for the period of a pendulum. The first step in your procedure will be to find an expression for a linear restoring force that causes the pendulum bob to return to its equilibrium position. You will then substitute the proportionality constant from that equation into the period for a mass and spring system in place of the force constant, k.

To start developing the equation for the restoring force for the pendulum, examine Figure 13.8. When you pull a pendulum bob to the side, you give it gravitational potential energy. Therefore, it is the force of gravity that causes the bob to return to its equilibrium position. If you keep the string taut while lifting the bob, it moves a distance d along the path of the arc of a circle as shown in the figure. The following steps will guide you through the development of the period of the pendulum.

- The restoring force must act along the path of the pendulum which is tangent to the arc. As you can see in the red triangle in the figure, the magnitude of the restoring force, F_R, is the component of the weight of the bob that is tangent to the arc. It is negative because it is in the direction that would reduce the distance, d.

$$F_R = -F_g \sin\theta$$
$$F_R = -mg \sin\theta$$

- Inspection of the green triangle reveals that you can write $\sin\theta$ in terms of the properties of the pendulum, that is, the length, ℓ of the pendulum string.

$$F_R = -mg\frac{x}{\ell}$$

- Rearrange the equation to combine the constants and isolate the variable, x.

$$F_R = -\frac{mg}{\ell}x$$

- The constants, $\frac{mg}{\ell}$, are performing the same function in this relationship as k in Hooke's law. Substitute $\frac{mg}{\ell}$ for k in the period of a mass and spring system and simplify the expression.

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$T = 2\pi\sqrt{\frac{m}{\frac{mg}{\ell}}}$$
$$T = 2\pi\sqrt{\cancel{m}\frac{\ell}{\cancel{m}g}}$$
$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

Did you notice any inconsistencies in this derivation? The sine function is not a linear function, so the force is not precisely a *linear* restoring force. When you substitute x/ℓ for $\sin\theta$, however, it appears to be a linear function but now the distance, x, is not identical to, d, the length of the path taken by the pendulum bob. Nevertheless, if the angle between the equilibrium line and the pendulum string is less than 15°, the error caused by these non-ideal conditions will be less than one percent. Thus the equation is valid for the period of a pendulum for small angles. Also, notice that the only property of the pendulum that affects its period is the length. The mass of the pendulum bob has no effect on the period.

Conceptual Problems

- Would a pendulum have the same period on the moon that it has on Earth? Explain.
- Would a mass and spring system have the same period on the moon that it has on Earth? Explain.
- Would a spring and mass system oscillate on the Space Station?
- Would a pendulum swing on the Space Station?

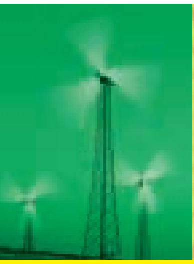
The period of a pendulum is the product of two pi and the square root of the quotient of length of the pendulum and the acceleration due to gravity.

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

Quantity	Symbol	SI unit
period of pendulum	T	s (seconds)
length of pendulum	ℓ	m (metres)
acceleration due to gravity	g	$\frac{\mathrm{m}}{\mathrm{s}^2}$ (metres per second squared)

Unit Analysis

$$\text{seconds} = \sqrt{\frac{\text{metres}}{\frac{\text{metres}}{\text{seconds}^2}}} \qquad \mathrm{s} = \sqrt{\frac{\mathrm{m}}{\frac{\mathrm{m}}{\mathrm{s}^2}}} = \sqrt{\cancel{\mathrm{m}}\,\frac{\mathrm{s}^2}{\cancel{\mathrm{m}}}} = \sqrt{\mathrm{s}^2} = \mathrm{s}$$



QUICK LAB

Measuring the Value of *g*, the Acceleration Due to Gravity

TARGET SKILLS

- Initiating and planning
- Performing and recording
- Analyzing and interpreting

You have probably measured the acceleration due to gravity using a ticker tape timer attached to a falling mass and with an Atwood's machine. The period of a pendulum offers a completely different way to measure g.

Assemble and measure the periods of pendulums of at least three very different lengths. When assembling the pendulums, ensure that the mass of the pendulum bob is much greater than the mass of the string. When taking data, measure the time required for at least 10 full cycles and then divide the time by the number of cycles. At what point within the pendulum is gravity effectively acting on the mass? Calculate the value of g using the periods and the pendulum lengths that you measured. Determine the percent deviation between your values of g and the accepted value of 9.81 m/s^2.

Analyzing and Concluding

1. Why is it important to make the mass of the pendulum bob significantly greater than the mass of the string?
2. Why should you measure the time for ten cycles rather than one cycle?
3. Where, within the pendulum bob does the force of gravity appear to act?
4. Which length, longest, middle, or shortest, gave the most accurate value of g?
5. What are the most likely sources of error in your determination of g using the pendulum?
6. What would be the most likely sources of error in the measurement of g
   - **(a)** using a falling mass and a ticker tape timer?
   - **(b)** using an Atwood's machine?

MODEL PROBLEM

(a) Find the period of a pendulum with a 2.45 kg bob and having a length of 1.36 m.

(b) By what amount would you have to increase the length in order to double the period?

Frame the Problem

- The only property of a pendulum that affects the period is the *length*.
- The relationship between *period* and *length* of a pendulum is *not* linear.

Identify the Goal

(a) the period, T of the pendulum

(b) the increase in length, $\Delta\ell$, of the pendulum

Variables and Constants

Known	Implied	Unknown	
$\ell_1 = 1.36\ \text{m}$	$g = 9.81\ \frac{\text{m}}{\text{s}^2}$	T_1	ℓ_2
$T_2 = 2T_1$		T_2	$\Delta\ell$

Strategy / Calculations

Write the equation for the period of a pendulum.

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

Substitute numerical values and solve.

$$T_1 = 2\pi\sqrt{\frac{1.36\ \cancel{\text{m}}}{9.81\frac{\cancel{\text{m}}}{\text{s}^2}}}$$

$$T_1 = 2\pi\sqrt{0.138634\ \text{s}^2}$$

$$T_1 = 2.33945\ \text{s}$$

$$T_1 \cong 2.34\ \text{s}$$

(a) The period is 2.34 s.

Double the first period to find the second.

$$T_2 = 2T_1$$
$$T_2 = 2(2.33945\ \text{s})$$
$$T_2 = 4.6789\ \text{s}$$
$$T_2 \cong 4.68\ \text{s}$$

Write the equation for the period of a pendulum and square both sides.

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

Then solve for length.

$$T^2\left(\frac{g}{4\pi^2}\right) = \cancel{4\pi^2}\left(\frac{\ell}{\cancel{g}}\right)\left(\frac{\cancel{g}}{\cancel{4\pi^2}}\right)$$

$$\ell = \frac{T^2 g}{4\pi^2}$$

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Substitute numerical values and solve.

$$\ell_2 = \frac{(4.6789\ \cancel{s})^2\,(9.81\ \frac{m}{\cancel{s^2}})}{4\pi^2}$$

$$\ell_2 = 5.44\ m$$

Subtract ℓ_1 from ℓ_2.

$$\Delta\ell = \ell_2 - \ell_1$$

$$\Delta\ell = 5.44\ m - 1.36\ m$$

$$\Delta\ell = 4.08\ m$$

(b) The *increase* in length must be 4.08 m.

Validate

All of the units cancel properly to give second for period and metres for length. Also, since the period is proportional to the square root of the length, you would expect that the increased length would have to be two squared or four times greater than the original length in order to double the period. ℓ_2 is exactly four times ℓ_1.

PRACTICE PROBLEMS

5. What is the period of a pendulum with a length of 0.45 m?

6. What must be the length of a pendulum to give it a period of 4.0 s?

7. If every swing of the pendulum in a clock causes the second hand to move an angle representing exactly one half a second, what must the length of the pendulum be to make the clock keep accurate time?

8. If a pendulum has a period of 0.36 s on Earth, what would its period be on the Moon?

13.1 Section Review

1. (a) Are all examples of periodic motion also examples of simple harmonic motion? Explain.

(b) Are all examples of simple harmonic motion also examples of periodic motion? Explain.

2. Explain the logic that was used to verify that the shadows of a marker moving with uniform circular motion is mathematically identical the motion of a mass on the end of a spring.

3. What must be true of a system to make it move with simple harmonic motion?

4. Explain the conditions for use of the equation, $T = 2\pi\sqrt{\frac{\ell}{g}}$, for the period of a pendulum.