11.1 Projectile Motion

SECTION OUTCOMES

- Describe qualitatively and quantitatively the path of a projectile.
- Analyze, predict, and explain projectile motion in terms of horizontal and vertical components.
- Design and conduct experiments to test the predictions about the motion of a projectile.

KEY TERMS

- trajectory
- projectile
- range
- parabola

Figure 11.1 After the water leaves the pipes in this fountain, the only forces acting on the water are gravity and air friction.

A tourist visiting Monte Carlo, Monaco, would probably stand and admire the beauty of the fountain shown in the photograph, and might even toss a coin into the fountain and make a wish. A physics student, however, might admire the symmetry of the water jets. He or she might estimate the highest point that the water reaches and the angle at which it leaves the fountain, and then mentally calculate the initial velocity the water must have in order to reach that height.

The student might then try to think of as many examples of this type of motion as possible. For example, a golf ball hit off the tee, a leaping frog, a punted football, and a show-jumping horse all follow the same type of path or **trajectory** as the water from a fountain. Any object given an initial thrust and then allowed to soar through the air under the force of gravity only is called a **projectile**. The horizontal distance that the projectile travels is called its **range**.

Air friction does, of course, affect the trajectory of a projectile and therefore the range of the projectile, but the mathematics needed to account for air friction is complex. You can learn a great deal about the trajectory of projectiles by neglecting friction, while keeping in mind that air friction will modify the actual motion.

You do not need to learn any new concepts in order to analyze and predict the motion of projectiles. All you need are data that will provide you with the velocity of the projectile at the moment it is launched and the kinematic equations for uniformly accelerated motion. You observed projectile motion in the Race

History Link

Calculating actual trajectories of artillery shells was an enormous task before the advent of electronic computers. Human experts required up to 20 h to do the job, even with mechanical calculators. ENIAC, the first electronic computer that stored its program instructions, was built in 1946 and could calculate the trajectory of an artillery shell in about 30 s. The vacuum tubes in ENIAC's processing unit required 174 kW of electric power, so the energy required to calculate one trajectory was comparable to the energy needed to actually fire the artillery shell!

to the Ground segment of the Multi-Lab and identified a feature of the motion that simplifies the analysis. The horizontal motion of the projectile does not influence the vertical motion, nor does the vertical motion affect the horizontal motion. *You can treat the motion in the two directions independently.* The following points will help you analyze all instances of projectile motion.

- Gravity is the only force influencing ideal projectile motion. (Neglect air friction.)
- Gravity affects only the vertical motion, so equations for uniformly accelerated motion apply.
- No forces affect horizontal motion, so equations for uniform motion apply.
- The horizontal and vertical motions are taking place during the same time interval, thus providing a link between the motion in these dimensions.

Projectiles Launched Horizontally

If you had taken a picture with a strobe light of your Race to the Ground lab, you would have obtained a photograph similar to the one in Figure 11.2. The ball on the right was given an initial horizontal velocity while, at the same moment, the ball on the left was dropped. As you can see in the photograph, the two balls were the same distance from the floor at any given time — the vertical motion of the two balls was identical. This observation verifies that horizontal motion does not influence vertical motion. Examine the following model problem to learn how to make use of this feature of projectile motion.

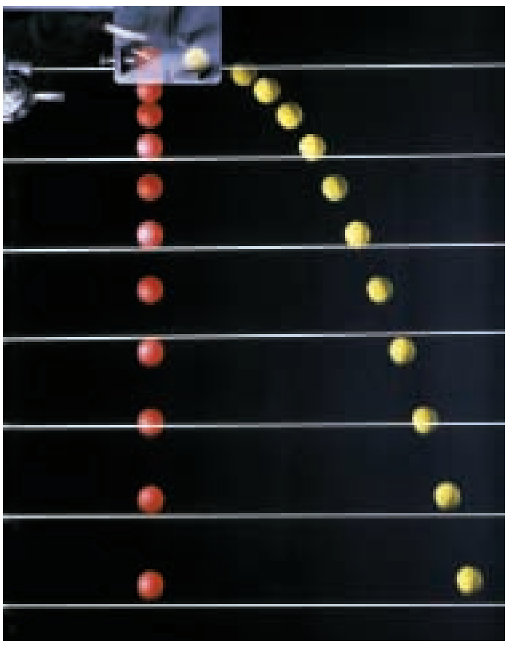
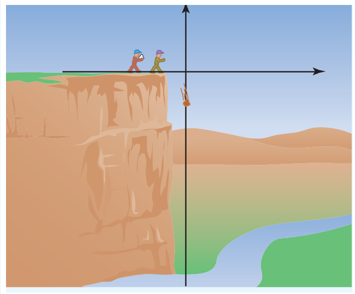


Figure 11.2 You can see that the balls are accelerating downward, because the distances they have travelled between flashes of the strobe light are increasing. If you inspected the horizontal motion of the ball on the right, you would find that it travelled the same horizontal distance between each flash of the strobe light.

MODEL PROBLEM

Analyzing a Horizontal Projectile

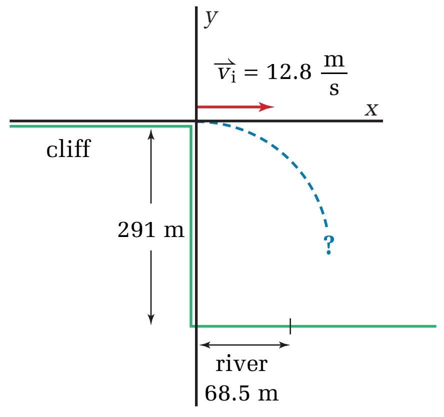
While hiking in the wilderness, you come to a cliff overlooking a river. A topographical map shows that the cliff is 291 m high and the river is 68.5 m wide at that point. You throw a rock directly forward from the top of the cliff, giving the rock a horizontal velocity of 12.8 m/s.



(a) Did the rock make it across the river?

(b) With what velocity did the rock hit the ground or water?

Frame the Problem



- Start to frame the problem by making a rough sketch of the cliff with a *coordinate system* superimposed on it. Write the initial conditions on the sketch.
- The rock *initially* has *no vertical velocity*. It falls, from *rest*, with the *acceleration due to gravity*. Since "down" was chosen as negative, the *acceleration* of the rock is *negative*. (Neglect air friction.)
- Since the coordinate system was placed at the top of the cliff, the *vertical component* of the *displacement* of the rock is *negative*.
- The *displacement* that the rock falls determines the *time interval* during which it falls, according to the kinematic equations.
- The rock moves *horizontally* with a *constant velocity* until it hits the ground or water at the end of the time interval.
- The *final velocity* of the rock at the instant before it hits the ground or water is the *vector sum* of the horizontal velocity and the final vertical velocity.
- Use x to represent the horizontal component of displacement and y for the vertical component of displacement. Use x and y subscripts to identify the horizontal and vertical components of the velocity.

Identify the Goal

(a) Whether the horizontal distance, Δx, travelled by the rock was greater than 68.5 m, the width of the river

(b) The final velocity, $\vec{v}_f$, of the rock the instant before it hit the ground

Variables and Constants

Known		Implied	Unknown
$\Delta y = -291$ m	river width = 68.5 m	$a_y = -9.81\ \frac{m}{s^2}$	Δx
$v_x = 12.8\ \frac{m}{s}$		$v_{iy} = 0.0\ \frac{m}{s}$	$\vec{v}_f$

Strategy

Find the time interval during which the rock was falling by using the kinematic equation that relates displacement, initial velocity, acceleration, and time interval. Note that the vertical component of the initial velocity is zero and solve for the time interval.

Calculations

$$\Delta y = v_{yi}\Delta t + \frac{1}{2}a\Delta t^2$$
$$\Delta y = \frac{1}{2}a\Delta t^2$$
$$\frac{2\Delta y}{a} = \Delta t^2$$
$$\Delta t = \sqrt{\frac{2\Delta y}{a}}$$

Insert numerical values and solve.

$$\Delta t = \sqrt{\frac{2(-291\ \cancel{\text{m}})}{-9.81\ \frac{\cancel{\text{m}}}{\text{s}^2}}}$$
$$\Delta t = \sqrt{59.327\ \text{s}^2}$$
$$\Delta t = \pm 7.7024\ \text{s}$$

Find the horizontal displacement of the rock by using the equation for uniform motion (constant velocity) that relates velocity, distance, and time interval. Solve for displacement.

$$v_x = \frac{\Delta x}{\Delta t}$$
$$\Delta x = v_x\Delta t$$

Use the time calculated above and initial velocity to calculate the horizontal distance travelled by the rock. Choose the positive value for time, since negative time has no meaning in this application.

$$\Delta x = \left(12.8\ \frac{\text{m}}{\cancel{\text{s}}}\right)(7.7024\ \cancel{\text{s}})$$
$$\Delta x = 98.591\ \text{m}$$
$$\Delta x \cong 98.6\ \text{m}$$

(a) Since the horizontal distance travelled by the rock (98.6 m) was much greater than the width of the river (68.5 m), the rock hit the ground on the far side of the river.

Find the vertical component of the final velocity by using the kinematic equation that relates initial velocity, final velocity, acceleration, and time.

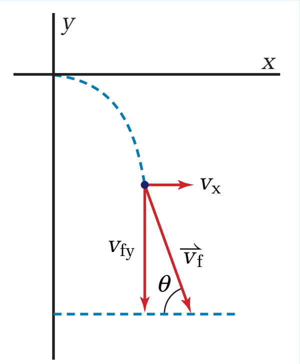
$$v_{fy} = v_{iy} + a\Delta t$$

Insert the numerical values and solve.

$$v_{fy} = 0.0\ \frac{\text{m}}{\text{s}} + \left(-9.81\ \frac{\text{m}}{\text{s}^2}\right)(7.7024\ \text{s})$$
$$v_{fy} = -75.561\ \frac{\text{m}}{\text{s}}$$

Use the Pythagorean theorem to find the magnitude of the resultant velocity.

$$|\vec{v}_f| = \sqrt{(v_x)^2 + (v_{fy})^2}$$
$$|\vec{v}_f| = \sqrt{\left(12.8\ \frac{\text{m}}{\text{s}}\right)^2 + \left(-75.561\ \frac{\text{m}}{\text{s}}\right)^2}$$
$$|\vec{v}_f| = \sqrt{5873.30\ \frac{\text{m}^2}{\text{s}^2}}$$
$$|\vec{v}_f| = 76.637\ \frac{\text{m}}{\text{s}}$$
$$|\vec{v}_f| \cong 76.6\ \frac{\text{m}}{\text{s}}$$



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Use trigonometry to find the angle that the rock made with the horizontal when it struck the ground.

$$\tan\theta = \frac{v_{fy}}{v_x}$$
$$\theta = \tan^{-1}\frac{v_{fy}}{v_x}$$
$$\theta = \tan^{-1}\frac{75.561\ \frac{m}{s}}{12.8\ \frac{m}{s}}$$
$$\theta = \tan^{-1} 5.9032$$
$$\theta = 80.385^\circ$$
$$\theta \cong 80.4^\circ$$

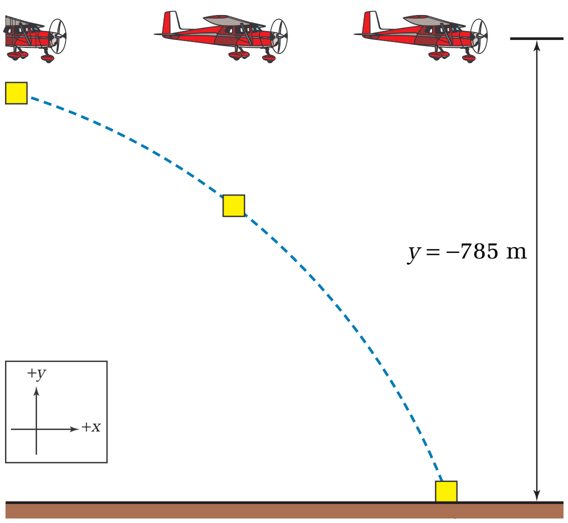
(b) The rock hit the ground with a velocity of 76.6 m/s at an angle of 80.4° with the horizontal.

Validate

The distance that the rock fell vertically was very large, so you would expect that the rock would be travelling very fast and that it would hit the ground at an angle that was nearly perpendicular to the ground. Both conditions were observed.

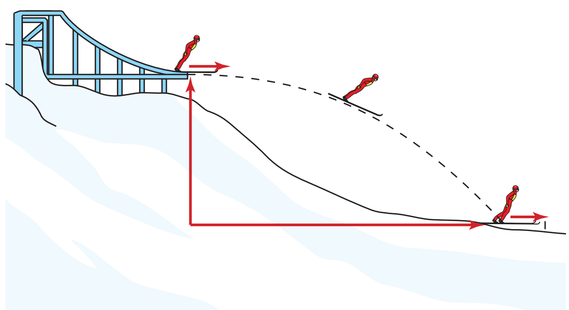
PRACTICE PROBLEMS

1. An airplane is dropping supplies to northern villages that are isolated by severe blizzards and cannot be reached by land vehicles. The airplane is flying at an altitude of 785 m and at a constant horizontal velocity of 53.5 m/s. At what horizontal distance before the drop point should the co-pilot drop the supplies so that they will land at the drop point? (Neglect air friction.)



2. A cougar is crouched on the branch of a tree that is 3.82 m above the ground. He sees an unsuspecting rabbit on the ground, sitting 4.12 m from the spot directly below the branch on which he is crouched. At what horizontal velocity should the cougar jump from the branch in order to land at the point at which the rabbit is sitting?

3. A skier leaves a jump with a horizontal velocity of 22.4 m/s. If the landing point is 78.5 m lower than the end of the ski jump, what horizontal distance did the skier jump? What was the skier's velocity when she landed? (Neglect air friction.)



4. An archer shoots an arrow toward a target, giving it a horizontal velocity of 70.1 m/s. If the target is 12.5 m away from the archer, at what vertical distance below the point of release will the arrow hit the target? (Neglect air friction.)

5. In a physics experiment, you are rolling a golf ball off a table. If the tabletop is 1.22 m above the floor and the golf ball hits the floor 1.52 m horizontally from the table, what was the initial velocity of the golf ball?

6. As you sit at your desk at home, your favourite autographed baseball rolls across a shelf at 1.0 m/s and falls 1.5 m to the floor. How far does it land from the base of the shelf?

7. A stone is thrown horizontally at 22 m/s from a canyon wall that is 55 m high. At what distance from the base of the canyon wall will the stone land?

8. A sharpshooter shoots a bullet horizontally over level ground with a velocity of 3.00×10^2 m/s. At the instant that the bullet leaves the barrel, its empty shell casing falls vertically and strikes the ground with a vertical velocity of 5.00 m/s.
   (a) How far does the bullet travel?
   (b) What is the vertical component of the bullet's velocity at the instant before it hits the ground?

Projectiles Launched at an Angle

Most projectiles, including living ones such as the playful dolphins in Figure 11.3, do not start their trajectory horizontally. Most projectiles, from footballs to frogs, start at an angle with the horizontal. Consequently, they have an initial velocity in both the horizontal and vertical directions. These trajectories are described mathematically as **parabolas**. The only additional step required to analyze the motion of projectiles launched at an angle is to determine the magnitude of the horizontal and vertical components of the initial velocity.

Mathematically, the path of any ideal projectile lies along a parabola. In the following investigation, you will develop some mathematical relationships that describe parabolas. Then, the model problems that follow will help you apply mathematical techniques for analyzing projectiles.

Figure 11.3 Dolphins have been seen jumping as high as 4.9 m from the surface of the water in a behaviour called a "breach."

INVESTIGATION 11-A

The Components of Projectile Motion

TARGET SKILLS

- Analyzing and interpreting
- Modelling concepts
- Communicating results

A heavy steel ball rolling up and down a ramp follows the same type of trajectory that a projectile follows. You will obtain a permanent record of the steel ball's path by placing a set of white paper and carbon paper in its path. You will then analyze the vertical and horizontal motion of the ball and find mathematical relationships that describe the path.

Problem

What patterns exist in the horizontal and vertical components of projectile velocity?

Hypothesis

Formulate a hypothesis about the relationships between time and the vertical distance travelled by the steel ball.

Equipment

- large sheet of plywood
- very heavy steel ball
- metre stick, graph paper, tape
- set of white paper and carbon paper (or pressure-sensitive paper)

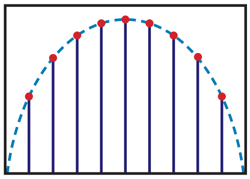
Procedure

1. Set up the apparatus as illustrated.
   CAUTION Wear impact-resistant safety goggles. Also, do not stand close to other people or equipment while doing these activities.

2. Practise rolling the steel ball up the slope at an angle, so that it follows a curved path that will fit the size of your set of white paper and carbon paper.
3. Tape the carbon paper and white paper onto the plywood so that, when the steel ball rolls over it, the carbon paper will leave marks on the white paper.
4. Roll the steel ball up the slope at an angle, as you practised, so that it will roll over the paper and leave a record of its path.
5. Remove the white paper from the plywood. Draw approximately nine or more equally spaced lines vertically through the trajectory.

Analyze and Conclude

1. Measure the vertical displacement in each segment of the path of the steel ball, as shown in the diagram.



2. Assuming that the motion of the ball was uniform in the horizontal direction, each equally spaced vertical line represents the same amount of time. Call it one unit of time.
3. Separate your data into two parts: (a) the period of time that the ball was rolling upward and (b) the period of time that the ball was rolling downward. For each set of data, make a graph of vertical-distance-versus-time units.
4. Use curve-straightening techniques to convert your graphs to straight lines. (See Skill Set 4.)
5. Write equations to describe your graphs.
6. Is the vertical motion of the steel ball uniform or uniformly accelerated?
7. How does it compare to the vertical motion of a freely falling object?
8. Was your hypothesis valid or invalid?
9. Is this lab an appropriate model for actual projectile motion? Explain why or why not.

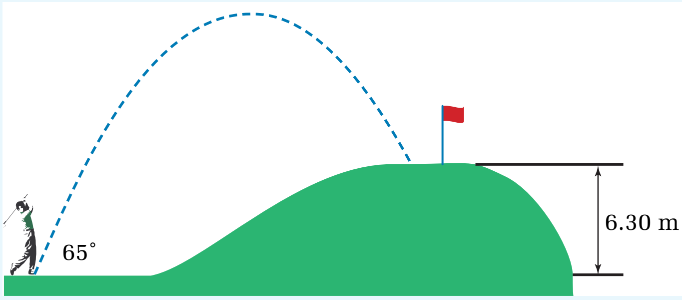
Analyzing Parabolic Trajectories

1. **A golfer hits the golf ball off the tee, giving it an initial velocity of 32.6 m/s at an angle of 65° with the horizontal. The green where the golf ball lands is 6.30 m higher than the tee, as shown in the illustration. Find**

(a) the time interval during which the golf ball was in the air

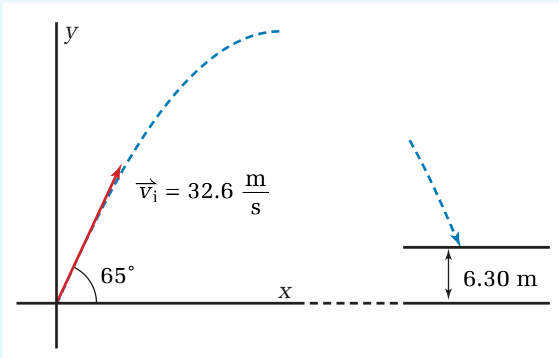
(b) the horizontal distance that it travelled

(c) the velocity of the ball just before it hit the ground (neglect air friction)



Frame the Problem

- Start to frame the problem by making a sketch that includes a *coordinate system*, the *initial conditions*, and all of the known information.
- The golf ball has a *positive initial velocity* in the *vertical* direction. It will rise and then fall according to the kinematic equations.
- The *vertical acceleration* of the golf ball is *negative* and has the *magnitude* of the acceleration due to *gravity*.
- The *time interval* is determined by the *vertical* motion. The *time interval ends* when the golf ball is at a height equal to the *height of the green*.
- The golf ball will be at the height of the green *twice*, once while it is *rising* and once while it is *falling*.
- Motion in the *horizontal* direction is *uniform*; that is, it has a *constant velocity*.
- The horizontal *displacement of the ball* depends on the *horizontal component* of the *initial velocity* and on the duration of the flight.



Identify the Goal

(a) The time interval, Δt, that the golf ball was in the air

(b) The horizontal distance, Δx, that the golf ball travelled

(c) The final velocity of the golf ball, $\vec{v}_f$

Variables and Constants

Known		Implied	Unknown	
$|\vec{v}_i| = 32.6\ \frac{m}{s}$	$\Delta y = 6.30$ m	$a_y = -9.81\ \frac{m}{s^2}$	Δt	$\vec{v}_f$
$\theta_i = 65°$			Δx	θ_f
			v_{ix}	v_{iy}

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Strategy	Calculations
Find the horizontal and vertical components of the initial velocity.	$v_{ix} = \lvert\vec{v}_i\rvert \cos\theta$ $v_{iy} = \lvert\vec{v}_i\rvert \sin\theta$ $v_{ix} = 32.6\ \frac{m}{s} \cos 65°$ $v_{iy} = 32.6\ \frac{m}{s} \sin 65°$ $v_{ix} = 13.78\ \frac{m}{s}$ $v_{iy} = 29.55\ \frac{m}{s}$
Find the time interval at which the ball is at a vertical position of 6.30 m by using the kinematic equation that relates displacement, initial velocity, acceleration, and the time interval. You cannot solve directly for the time interval, because you have a quadratic equation.	$\Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2$
Substitute in the numerical values.	$6.30\ m = 29.55\ \frac{m}{s}\Delta t + \frac{1}{2}\left(-9.81\ \frac{m}{s^2}\right)\Delta t^2$
Rearrange the equation into the general form of a quadratic equation and solve using the quadratic formula $\Delta t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.	$4.905\Delta t^2 - 29.55\Delta t + 6.30 = 0$ $\Delta t = \frac{29.55 \pm \sqrt{(29.55)^2 - 4(4.905)(6.30)}}{2(4.905)}$ $\Delta t = \frac{29.55 \pm \sqrt{749.597}}{9.81}$ $\Delta t = 0.2213\ s\ (or)\ 5.803\ s$ $\Delta t \cong 5.8\ s$

(a) The smaller value is the time that the ball reached a height of 6.30 m when it was rising. The golf ball hit the green 5.8 s after it was hit off the tee.

Use 5.803 s and the equation for constant velocity to determine the horizontal distance travelled by the golf ball.	$v = \frac{\Delta x}{\Delta t}$ $\Delta x = v\Delta t$ $\Delta x = \left(13.78\ \frac{m}{\cancel{s}}\right)(5.803\ \cancel{s})$ $\Delta x = 79.965\ m$ $\Delta x \cong 8.0 \times 10^1\ m$

(b) The golf ball travelled 80 m in the horizontal direction.

Find the vertical component of the final velocity by using the kinematic equation that relates the initial and final velocities to the acceleration and the time interval.	$v_{fy} = v_{iy} + a_y\Delta t$ $v_{fy} = 29.55\ \frac{m}{s} + \left(-9.81\ \frac{m}{s^{\cancel{2}}}\right)(5.803\ \cancel{s})$ $v_{fy} = -27.38\ \frac{m}{s}$

Use the Pythagorean theorem to find the magnitude of the final velocity.

$$|\vec{v}_f| = \sqrt{\left(13.78\ \frac{m}{s}\right)^2 + \left(-27.38\ \frac{m}{s}\right)^2}$$

$$|\vec{v}_f| = \sqrt{939.55\ \frac{m^2}{s^2}}$$

$$|\vec{v}_f| = 30.65\ \frac{m}{s}$$

$$|\vec{v}_f| \cong 31\ \frac{m}{s}$$

Use trigonometry to find the angle that the final velocity makes with the horizontal.

$$\tan\theta = \left(\frac{v_{fy}}{v_x}\right)$$

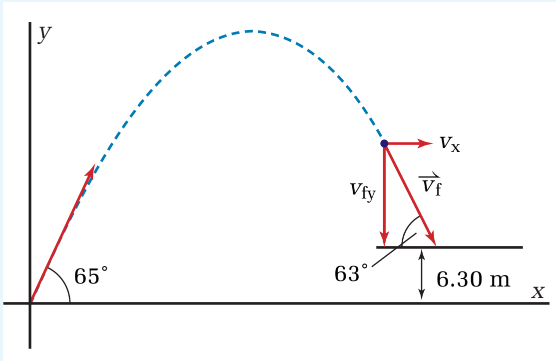
$$\theta = \tan^{-1}\left(\frac{v_{fy}}{v_x}\right)$$

$$\theta = \tan^{-1}\frac{|-27.38\ \frac{m}{s}|}{|13.78\ \frac{m}{s}|}$$

$$\theta = \tan^{-1} 1.9869$$

$$\theta = 63.28^\circ$$

$$\theta \cong 63^\circ$$



(c) The final velocity of the golf ball just before it hit the ground was 31 m/s at 63° with the horizontal.

Validate

Since the golf ball hit the ground at a level slightly higher than the level at which it started, you would expect the final velocity to be slightly smaller than the initial velocity and the angle to be a little smaller than the initial angle. These results were obtained. All of the units cancelled properly.

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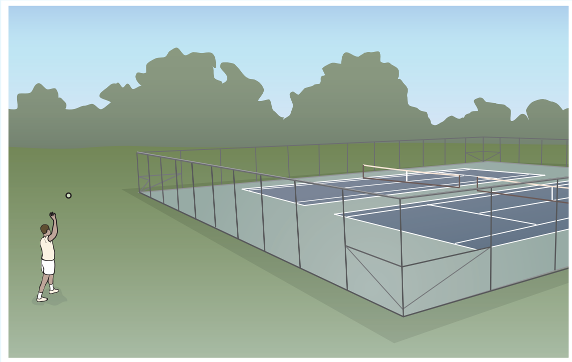
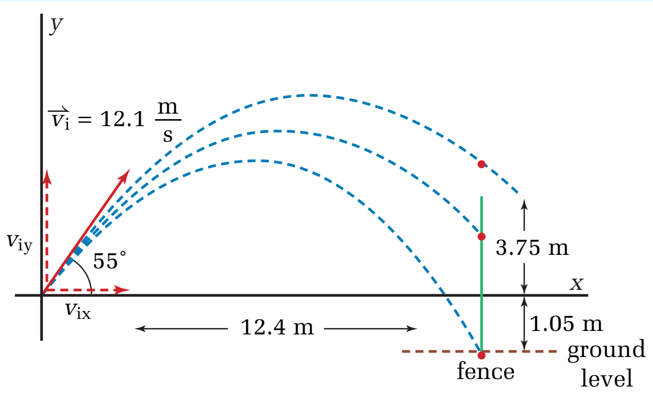
2. You are playing tennis with a friend on tennis courts that are surrounded by a 4.8 m fence. Your opponent hits the ball over the fence and you offer to retrieve it. You find the ball at a distance of 12.4 m on the other side of the fence. You throw the ball at an angle of 55.0° with the horizontal, giving it an initial velocity of 12.1 m/s. The ball is 1.05 m above the ground when you release it. Did the ball go over the fence, hit the fence, or hit the ground before it reached the fence? (Ignore air friction.)



Frame the Problem

- Make a sketch of the *initial conditions* and the three options listed in the question.
- Choose the *origin* of the coordinate system to be at the point at which the *ball left your hand*.
- The equations for *uniformly accelerated motion* apply to the *vertical* motion.
- The definition for *constant velocity* applies to the *horizontal* motion.
- Because the *x*-axis is above ground level, you will have to determine where the top of the fence is relative to the *x*-axis.
- The *time interval* is the *link* between the *vertical* motion and the *horizontal* motion. Finding the time interval required for the ball to reach the position of the fence will allow you to determine the *height* of the ball when it reaches the fence.



Identify the Goal

Whether the ball went over the fence, hit the fence, or hit the ground before reaching the fence

Variables and Constants

Known		Implied	Unknown	
$\lvert\vec{v}_i\rvert = 12.1\ \frac{m}{s}$	$\Delta x = 12.4$ m	$a_y = -9.81\ \frac{m}{s^2}$	Δt	$\vec{v}_{iy}$
$\theta = 55°$	$h = 4.8$ m		$\vec{v}_{ix}$	Δy

Strategy

Find the *x*- and *y*-components of the initial velocity.

Calculations

$$v_{ix} = \lvert\vec{v}_i\rvert \cos\theta \qquad v_{iy} = \lvert\vec{v}_i\rvert \sin\theta$$

$$v_{ix} = 12.1\ \frac{m}{s} \cos 55° \qquad v_{iy} = 12.1\ \frac{m}{s} \sin 55°$$

$$v_{ix} = 6.940\ \frac{m}{s} \qquad v_{iy} = 9.912\ \frac{m}{s}$$

To find the time interval, use the equation for the definition of constant velocity and the data for motion in the horizontal direction.

$$v_x = \frac{\Delta x}{\Delta t}$$

$$\Delta t = \frac{\Delta x}{v_x}$$

$$\Delta t = \frac{12.4\ \cancel{\text{m}}}{6.940\ \frac{\cancel{\text{m}}}{\text{s}}}$$

$$\Delta t = 1.787\ \text{s}$$

To find the height of the ball at the time that it reaches the fence, use the kinematic equation that relates displacement, acceleration, initial velocity, and time interval.

$$\Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2$$

$$\Delta y = \left(9.912\ \frac{\text{m}}{\cancel{\text{s}}}\right)(1.787\ \cancel{\text{s}}) + \frac{1}{2}\left(-9.81\ \frac{\text{m}}{\text{s}^{\cancel{2}}}\right)(1.787\ \cancel{\text{s}})^2$$

$$\Delta y = 2.05\ \text{m}$$

Determine the position of the top of the fence in the chosen coordinate system.

$$y_{\text{fence}} = h - y_{\text{ground to } x\text{–axis}}$$

$$y_{\text{fence}} = 4.8\ \text{m} - 1.05\ \text{m}$$

$$y_{\text{fence}} = 3.75\ \text{m}$$

The ball hit the fence. The fence is 3.75 m above the horizontal axis of the chosen coordinate system, but the ball was only 2.05 m above the horizontal axis when it reached the fence.

Validate

The units all cancel correctly. The time of flight (about 1.8 s) and the height of the ball (about 2 m) are reasonable values.

PRACTICE PROBLEMS

9. While hiking in the wilderness, you come to the top of a cliff that is 60.0 m high. You throw a stone from the cliff, giving it an initial velocity of 21 m/s at 35° above the horizontal. How far from the base of the cliff does the stone land?

10. A batter hits a baseball, giving it an initial velocity of 41 m/s at 47° above the horizontal. It is a home run, and the ball is caught by a fan in the stands. The vertical component of the velocity of the ball when the fan caught it was –11 m/s. How high is the fan seated above the field?

11. During baseball practice, you go up into the bleachers to retrieve a ball. You throw the ball back into the playing field at an angle of 42° above the horizontal, giving it an initial velocity of 15 m/s. If the ball is 5.3 m above the level of the playing field when you throw it, what will be the velocity of the ball when it hits the ground of the playing field?

12. Large insects such as locusts can jump as far as 75 cm horizontally on a level surface. An entomologist analyzed a photograph and found that the insect's launch angle was 55°. What was the insect's initial velocity?

You have learned to make predictions about projectile motion by doing calculations, but can you make any predictions about patterns of motion without doing calculations? In the following Quick Lab, you will make and test some qualitative predictions.

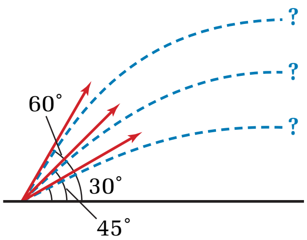
QUICK LAB

Maximum Range of a Projectile

TARGET SKILLS
- Initiating and planning
- Predicting
- Performing and recording

Football punters try to maximize “hang time” to give their teammates an opportunity to rush downfield while the ball is in the air. Small variations in the initial velocity, especially the angle, make the difference between a great kick and good field position for the opposition.

What launch angle above the horizontal do you predict would maximize the range of an ideal projectile? Make a prediction and then, if your school has a projectile launcher, test your prediction by launching the same projectile several times at the same speed, but at a variety of different angles. If you do not have a projectile launcher, try to devise a system that will allow you to launch a projectile consistently with the same speed but at different angles. Carry out enough trials so you can be confident that you have found the launch angle that gives the projectile the longest range. Always consult with your teacher before using a launch system.



Analyze and Conclude

1. What effect do very large launch angles have on the following quantities?
   (a) maximum height
   (b) vertical velocity component
   (c) horizontal velocity component
   (d) range
2. What effect do very small launch angles have on the above quantities?
3. Did you see any patterns in the relationship between the launch angle and the range of the projectile? If so, describe these patterns.
4. How well did your experimental results match your prediction?
5. What factors might be causing your projectile to deviate from the ideal?
6. Suppose your experimental results were quite different from your prediction. In which number would you place more confidence, your theoretical prediction or your experimental results? Why?

Symmetrical Trajectories

If a projectile lands at exactly the same level from which it was launched and air friction is neglected, the trajectory is a perfectly symmetrical parabola, as shown in Figure 11.4. You can derive some general relationships that apply to all symmetrical trajectories and use them to analyze these trajectories. Follow the steps in the next series of derivations to see how to determine the time of flight, the range, and the maximum height for projectiles that have symmetrical trajectories.

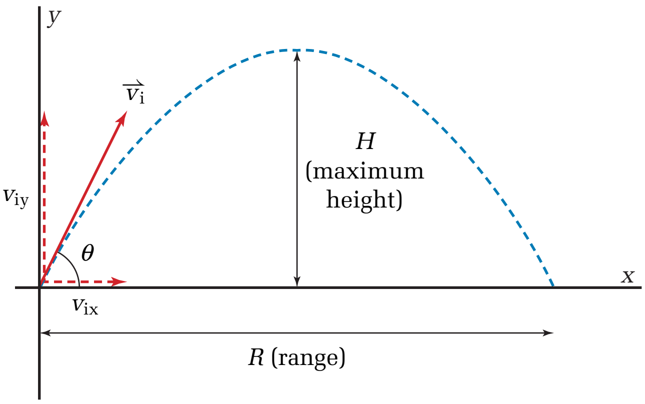


Figure 11.4 The maximum height, *H*, and the range, *R*, as well as the time of flight, *T*, are functions of the initial velocity, $\vec{v}_i$, the angle, θ, and the acceleration due to gravity, *g*.

Time of flight

- The time of flight ends when the projectile hits the ground. Since the height of the projectile is zero when it hits the ground, you can express this position as $\Delta y = 0$. Write the kinematic equation for vertical displacement and set $\Delta y = 0$.

$$\Delta y = v_{iy}\Delta t + \frac{1}{2}a\Delta t^2$$
$$0 = v_{iy}\Delta t + \frac{1}{2}a\Delta t^2$$

- Write the vertical component of the velocity in terms of the initial velocity and the angle θ. Then, substitute the expression into the equation above. Also, substitute $-g$ for the acceleration, a.

$$v_{iy} = |\vec{v}_i| \sin\theta$$
$$0 = (|\vec{v}_i| \sin\theta)\Delta t + \frac{1}{2}(-g)\Delta t^2$$

- Rearrange the equation to put the zero on the right-hand side and factor out a Δt.

$$\frac{1}{2}g\Delta t^2 - |\vec{v}_i|\Delta t \sin\theta = 0$$
$$\Delta t\left(\frac{1}{2}g\Delta t - |\vec{v}_i| \sin\theta\right) = 0$$

- If either factor is zero, the equation above is satisfied. Write the two solutions.

$$\Delta t = 0 \quad \text{or} \quad \frac{1}{2}g\Delta t = |\vec{v}_i| \sin\theta$$
$$\Delta t = \frac{2|\vec{v}_i| \sin\theta}{g}$$

- $\Delta t = 0$ represents the instant that the projectile was launched. Therefore, the second expression represents the time of flight, T, that the projectile spent in the air before it landed. Since T is a scalar, write the initial velocity without a vector symbol.

$$T = \frac{2v_i \sin\theta}{g}$$

Range

- The range is the horizontal distance that the projectile has travelled when it hits the ground. Write the equation for displacement in the horizontal direction.

$$\Delta x = v_{ix}\Delta t$$

- Write the expression for the horizontal component of velocity in terms of the initial velocity and the launch angle θ. Substitute this expression into the equation for the displacement above.

$$v_{ix} = |\vec{v}_i| \cos\theta$$
$$\Delta x = (|\vec{v}_i| \cos\theta)\Delta t$$

- Since the projectile is at the endpoint of its range, R, when $\Delta t = T$, substitute the expression for T into the equation and simplify. Since R is always in one dimension, omit the vector symbol for the initial velocity.

$$R = \frac{(v_i \cos\theta)(2v_i \sin\theta)}{g}$$
$$R = \frac{v_i^2 2\sin\theta\cos\theta}{g}$$

- Write the trigonometric identity for $2\sin\theta\cos\theta$ and substitute the simpler form into the equation.

$$2\sin\theta\cos\theta = \sin 2\theta$$
$$R = \frac{v_i^2 \sin 2\theta}{g}$$

Maximum height

- As a projectile rises, it slows its upward motion, stops, and then starts downward. Therefore, at its maximum height, its vertical component of velocity is zero. Write the kinematic equation that relates initial and final velocities, acceleration, and displacement and solve for displacement, Δy.

$$v_{fy}^2 = v_{iy}^2 + 2a\Delta y$$
$$0 = v_{iy}^2 + 2a\Delta y$$
$$2a\Delta y = -v_{iy}^2$$
$$\Delta y = \frac{-v_{iy}^2}{2a}$$

- Substitute the expression for initial vertical velocity in terms of the initial velocity and the launch angle, θ. Substitute $-g$ for a. Now Δy is the maximum height, H. Since H is always in one dimension, omit the vector symbol for the initial velocity.

$$H = \frac{-(|\vec{v}_i| \sin\theta)^2}{2(-g)}$$
$$H = \frac{v_i^2 \sin^2\theta}{2g}$$

These three relationships — time of flight, range, and the maximum height — allow you to make important predictions about projectile motion without performing calculations. For example, you can determine the launch angle that will give you the maximum range by inspecting the equation for range. Study the logic of the following steps.

- Inspect the equation for range.

$$R = \frac{v_i^2 \sin 2\theta}{g}$$

- For a given initial velocity on the surface of Earth, the only variable is θ. Therefore, the term "$\sin 2\theta$" determines the maximum range. The largest value that the sine of any angle can achieve is 1.

$$\sin 2\theta = 1$$
$$R_{max} = \frac{v_i^2(1)}{g}$$

- For what angle, θ, is $\sin 2\theta = 1$? Recall that the angle for which the sine is 1 is 90°. Use this information to find θ.

$$\sin 2\theta = 1$$
$$2\theta = \sin^{-1} 1$$
$$2\theta = 90^\circ$$
$$\theta = 45^\circ$$

For any symmetrical trajectory, neglecting air friction, the launch angle that yields the greatest range is 45˚. Draw some conclusions of your own by answering the questions in the Conceptual Problems that follow.

Conceptual Problems

- Examine the equation for maximum height. For a given initial velocity, what launch angle would give a projectile the greatest height? What would be the shape of its trajectory?
- Examine the equation for time of flight. For a given initial velocity, what launch angle would give a projectile the greatest time of flight? Would this be a good angle for a football punter? Why?
- Consider the equation for range and a launch angle of 30˚. What other launch angle would yield a range exactly equal to that of the range for an angle of 30˚?
- Find another pair of launch angles (in addition to your answer to the above question) that would yield identical ranges.
- The acceleration due to gravity on the Moon is roughly one sixth of that on Earth ($g_{\text{moon}} = \frac{1}{6}g$). For a projectile with a given initial velocity, determine the time of flight, range, and maximum height on the Moon relative to those values on Earth.
- The general equation for a parabola is $y = \text{A}x^2 + \text{B}x + \text{C}$, where A, B, and C are constants. Start with the following equations of motion for a projectile and develop one equation in terms of Δx and Δy by eliminating Δt. Show that the resulting equation, in which Δy is a function of Δx, describes a parabola. Note that the values for the initial velocity (v_{i}) and launch angle (θ) are constants for a given trajectory.

$$\Delta x = v_{\text{i}}\Delta t \cos\theta$$
$$\Delta y = v_{\text{i}}\Delta t \sin\theta - \frac{1}{2}g\Delta t^2$$

MODEL PROBLEM

Analyzing a Kickoff

A player kicks a football for the opening kickoff. He gives the ball an initial velocity of 29 m/s at an angle of 69˚ with the horizontal. Neglecting friction, determine the ball's maximum height, hang time, and range.

Frame the Problem

- A football field is level, so the trajectory of the ball is a *symmetrical parabola*.
- You can use the equations that were developed for symmetrical trajectories.
- "Hang time" is the time of flight of the ball.

continued

continued from previous page

Identify the Goal

The maximum height, H, of the football
The time of flight, T, of the football
The range, R, of the football

Variables and Constants

Known	**Implied**	**Unknown**
$\lvert\vec{v}_i\rvert = 29\ \frac{\text{m}}{\text{s}}$	$g = 9.81\ \frac{\text{m}}{\text{s}^2}$	H
$\theta = 69^\circ$		T
		R

Strategy / Calculations

Use the equation for the maximum height of a symmetrical trajectory.

$$H = \frac{v_i^2 \sin^2\theta}{2g}$$

Substitute the numerical values and solve.

$$H = \frac{\left(29\ \frac{\text{m}}{\text{s}}\right)^2 (\sin 69^\circ)^2}{2\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)}$$

$$H = \frac{\left(841\ \frac{\text{m}^{\cancel{2}}}{\cancel{\text{s}^2}}\right)(0.871\ 57)}{19.62\ \frac{\cancel{\text{m}}}{\cancel{\text{s}^2}}}$$

$$H = 37.359\ \text{m}$$

$$H \cong 37\ \text{m}$$

The maximum height the football reached was 37 m.

Use the equation for the time of flight of a symmetrical trajectory.

$$T = \frac{2v_i \sin\theta}{g}$$

Substitute the numerical values and solve.

$$T = \frac{2\left(29\ \frac{\text{m}}{\text{s}}\right)(\sin 69^\circ)}{\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)}$$

$$T = \frac{\left(58\ \frac{\cancel{\text{m}}}{\cancel{\text{s}}}\right)(0.933\ 58)}{9.81\ \frac{\cancel{\text{m}}}{\text{s}^{\cancel{2}}}}$$

$$T = 5.5196\ \text{s}$$

$$T \cong 5.5\ \text{s}$$

The time of flight, or hang time, of the football was 5.5 s.

Use the equation for the range of a symmetrical trajectory.

$$R = \frac{v_i^2 \sin 2\theta}{g}$$

Substitute the numerical values and solve.

$$R = \frac{\left(29\ \frac{\text{m}}{\text{s}}\right)^2 (\sin 2(69^\circ))}{9.81\ \frac{\text{m}}{\text{s}^2}}$$

$$R = \frac{\left(841\ \frac{\text{m}^{\cancel{2}}}{\cancel{\text{s}^2}}\right)(0.669\ 13)}{9.81\ \frac{\cancel{\text{m}}}{\cancel{\text{s}^2}}}$$

$$R = 57.3637\ \text{m}$$

$$R \cong 57\ \text{m}$$

The football travelled 57 m.

Validate

All of the values are reasonable for a football kickoff. In every case, the units cancel properly to give metres for the range and maximum height and seconds for the time of flight, or hang time.

PRACTICE PROBLEMS

13. A circus stunt person was launched as a human cannon ball over a Ferris wheel. His initial velocity was 24.8 m/s at an angle of 55°. (Neglect friction)

(a) Where should the safety net be positioned?

(b) If the Ferris wheel was placed halfway between the launch position and the safety net, what is the maximum height of the Ferris wheel over which the stunt person could travel?

(c) How much time did the stunt person spend in the air?

14. You want to shoot a stone with a slingshot and hit a target on the ground 14.6 m away. If you give the stone an initial velocity of 12.5 m/s, neglecting friction, what must be the launch angle in order for the stone to hit the target? What would be the maximum height reached by the stone? What would be its time of flight?

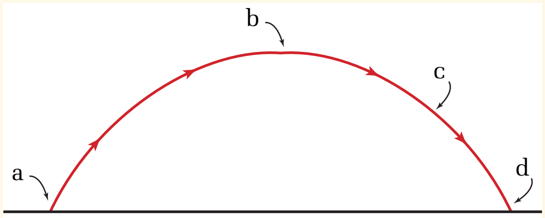
11.1 Section Review

1. K/U Projectiles travel in two dimensions at the same time. Why is it possible to apply kinematic equations for one dimension to projectile motion?

2. K/U How does the analysis of projectiles launched at an angle differ from the analysis of projectiles launched horizontally?

3. C Explain why time is a particularly significant parameter when analyzing projectile motion.

4. C What can you infer about the velocity at each labelled point on the trajectory in this diagram?



5. C Imagine that you are solving a problem in projectile motion in which you are asked to find the time at which a projectile reaches a certain vertical position. When you solve the problem, you find two different positive values for time that both satisfy the conditions of the problem. Explain how this result is not only possible, but also logical.

6. K/U What properties of projectile motion must you apply when deriving an equation for the maximum height of a projectile?

7. K/U What properties of projectile motion must you apply when deriving an equation for the range of a projectile?

8. I Suppose you knew the maximum height reached by a projectile. Could you find its launch angle from this information alone? If not, what additional information would be required?

CAREERS IN PHYSICS

Attacking Slime, Online! Tackling Biofilms with Computers

Bonnie Quinn

You may not know what a biofilm is, but chances are your dentist has scraped one off your teeth, or you've slipped on one while walking on the rocks in a stream or river. Biofilms form when bacteria settle on surfaces in aqueous environments, divide and increase in number, and surround themselves with a slimy, glue-like coating of polymeric material. Biofilms can form on any surface exposed to bacteria and some amount of water, and once they form, they can be at least 500 times more resistant to antibacterial agents than planktonic (isolated) bacteria moving around in the air or water.

Once it has anchored itself to a surface, a biofilm can do a lot of damage. They can corrode metal pipes, clog drains and water filters, cause tooth decay, contaminate food, and contaminate and lead to the rejection of medical implants like cardiac pacemakers. Given this undesirable list, it's no wonder that biofilm research is on the cutting edge of scientific research today. Bonnie Quinn, a computer programmer at St. Francis Xavier University (St.FX) in Antigonish, Nova Scotia, is one of many computer programmers, engineers, and scientists working to solve the problems posed by biofilms.

Quinn's favourite subject in high school was math. "It was easy," she says. "It made sense." When she left high school, Quinn took her love of math with her to the University of Toronto, where she studied math and computer science. After teaching high school in Ontario, she moved to Antigonish and started working at St.FX. For the last 17 years, Quinn has worked with St.FX physicist David Pink, designing, writing, and testing computer software to simulate mathematical models of complex systems found in nature.

Pink says that computer simulation of mathematical models is "one of the hottest ways to gain insight into the structure and dynamics of complex systems." These complex systems could involve bacteria defending themselves in hostile watery environments, fat globules in milk combining to form cheese, or water-soluble proteins attaching themselves to mitochondrial membranes. Quinn converts Pink's mathematics into a computer code. Once she has designed a code that correctly describes the logic of Pink's model, she runs the program, sometimes for months, on computers to generate statistics. If the model's statistics replicate the results gained in actual experiments, it is considered a good model.

In addition to her work with the physics department at St.FX, Quinn develops software for researchers in psychology and earth sciences. "If the problems I had to work on were tedious," she says, "I wouldn't stay with this work, but I've always been given new and interesting problems to work on, and the independence to figure them out by myself." In the summer months, Quinn trains some of the brightest students at the university. She says, "It's great to see them learn how to program and then actually do it all — design the mathematical model of the physics *and* write the computer program."

Going Further

1. For students who are interested in science and computers, Quinn recommends taking a course in computer programming. She says, "It can be a very creative process, like writing an essay. Until you understand how you can tell the machine what to do, you don't really understand how they work." Check out the computer programming courses offered in your high school, or at the local community college or university.